

Deep Network with Approximation Error Being Reciprocal of Width to Power of Square Root of Depth

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Keywords: Exponential Convergence, Curse of Dimensionality, Deep Neural Network, Floor and ReLU Activation Functions, Continuous Function.

Abstract

A new network with super approximation power is introduced. This network is built with Floor ($\lfloor x \rfloor$) or ReLU ($\max\{0, x\}$) activation function in each neuron and hence we call such networks Floor-ReLU networks. For any hyper-parameters $N \in \mathbb{N}^+$ and $L \in \mathbb{N}^+$, it is shown that Floor-ReLU networks with width $\max\{d, 5N + 13\}$ and depth $64dL + 3$ can uniformly approximate a Hölder function f on $[0, 1]^d$ with an approximation error $3\lambda d^{\alpha/2} N^{-\alpha\sqrt{L}}$, where $\alpha \in (0, 1]$ and λ are the Hölder order and constant, respectively. More generally for an arbitrary continuous function f on $[0, 1]^d$ with a modulus of continuity $\omega_f(\cdot)$, the constructive approximation rate is $\omega_f(\sqrt{d} N^{-\sqrt{L}}) + 2\omega_f(\sqrt{d}) N^{-\sqrt{L}}$. As a consequence, this new class of networks overcomes the curse of

34 dimensionality in approximation power when the variation of $\omega_f(r)$ as $r \rightarrow 0$ is mod-
 35 erate (e.g., $\omega_f(r) \lesssim r^\alpha$ for Hölder continuous functions), since the major term to be
 36 considered in our approximation rate is essentially \sqrt{d} times a function of N and L
 37 independent of d within the modulus of continuity.

38 1 Introduction

39 Recently, there has been a large number of successful real-world applications of deep
 40 neural networks in many fields of computer science and engineering, especially for
 41 large-scale and high-dimensional learning problems. Understanding the approximation
 42 capacity of deep neural networks has become a fundamental research direction for re-
 43 vealing the advantages of deep learning compared to traditional methods. This paper
 44 introduces new theories and network architectures achieving root exponential conver-
 45 gence and avoiding the curse of dimensionality simultaneously for (Hölder) continuous
 46 functions with an explicit error bound in deep network approximation, which might
 47 be two foundational laws supporting the application of deep network approximation in
 48 large-scale and high-dimensional problems. The approximation results here are quan-
 49 titative and apply to networks with essentially arbitrary width and depth. These results
 50 suggest considering Floor-ReLU networks as a possible alternative to ReLU networks
 51 in deep learning.

52 Deep ReLU networks with width $\mathcal{O}(N)$ and depth $\mathcal{O}(L)$ can achieve the approxi-
 53 mation rate $\mathcal{O}(N^{-L})$ for polynomials on $[0, 1]^d$ (Lu et al., 2020) but it is not true for gen-
 54 eral functions, e.g., the (nearly) optimal approximation rates of deep ReLU networks for
 55 a Lipschitz continuous function and a C^s function f on $[0, 1]^d$ are $\mathcal{O}(\sqrt{d}N^{-2/d}L^{-2/d})$
 56 and $\mathcal{O}(\|f\|_{C^s}N^{-2s/d}L^{-2s/d})$ (Shen et al., 2019b; Lu et al., 2020), respectively. The
 57 limitation of ReLU networks motivates us to explore other types of network architec-
 58 tures to answer our curiosity on deep networks: Do deep neural networks with arbi-
 59 trary width $\mathcal{O}(N)$ and arbitrary depth $\mathcal{O}(L)$ admit an exponential approximation rate
 60 $\mathcal{O}(\omega_f(N^{-L^\eta}))$ for some constant $\eta > 0$ for a generic continuous function f on $[0, 1]^d$
 61 with a modulus of continuity $\omega_f(\cdot)$?

62 To answer this question, we introduce the Floor-ReLU network, which is a fully
 63 connected neural network (FNN) built with either Floor ($\lfloor x \rfloor$) or ReLU ($\max\{0, x\}$)
 64 activation function¹ in each neuron. Mathematically, if we let $N_0 = d$, $N_{L+1} = 1$,
 65 and N_ℓ be the number of neurons in ℓ -th hidden layer of a Floor-ReLU network for
 66 $\ell = 1, 2, \dots, L$, then the architecture of this network with input x and output $\phi(x)$ can be

¹Our results can be easily generalized to Ceiling-ReLU networks, namely, feed-forward neural net-
 works with either Ceiling ($\lceil x \rceil$) or ReLU ($\max\{0, x\}$) activation function in each neuron.

67 described as

$$68 \quad \mathbf{x} = \tilde{\mathbf{h}}_0 \xrightarrow{\mathbf{W}_0, \mathbf{b}_0} \mathbf{h}_1 \xrightarrow{\sigma \text{ or } \lfloor \cdot \rfloor} \tilde{\mathbf{h}}_1 \quad \dots \quad \xrightarrow{\mathbf{W}_{L-1}, \mathbf{b}_{L-1}} \mathbf{h}_L \xrightarrow{\sigma \text{ or } \lfloor \cdot \rfloor} \tilde{\mathbf{h}}_L \xrightarrow{\mathbf{W}_L, \mathbf{b}_L} \mathbf{h}_{L+1} = \phi(\mathbf{x}),$$

69 where $\mathbf{W}_\ell \in \mathbb{R}^{N_{\ell+1} \times N_\ell}$, $\mathbf{b}_\ell \in \mathbb{R}^{N_{\ell+1}}$, $\mathbf{h}_{\ell+1} := \mathbf{W}_\ell \cdot \tilde{\mathbf{h}}_\ell + \mathbf{b}_\ell$ for $\ell = 0, 1, \dots, L$, and $\tilde{\mathbf{h}}_{\ell,n}$ is equal
70 to $\sigma(\mathbf{h}_{\ell,n})$ or $\lfloor \mathbf{h}_{\ell,n} \rfloor$ for $\ell = 1, 2, \dots, L$ and $n = 1, 2, \dots, N_\ell$, where $\mathbf{h}_\ell = (\mathbf{h}_{\ell,1}, \dots, \mathbf{h}_{\ell,N_\ell})$
71 and $\tilde{\mathbf{h}}_\ell = (\tilde{\mathbf{h}}_{\ell,1}, \dots, \tilde{\mathbf{h}}_{\ell,N_\ell})$ for $\ell = 1, 2, \dots, L$. See Figure 1 for an example.

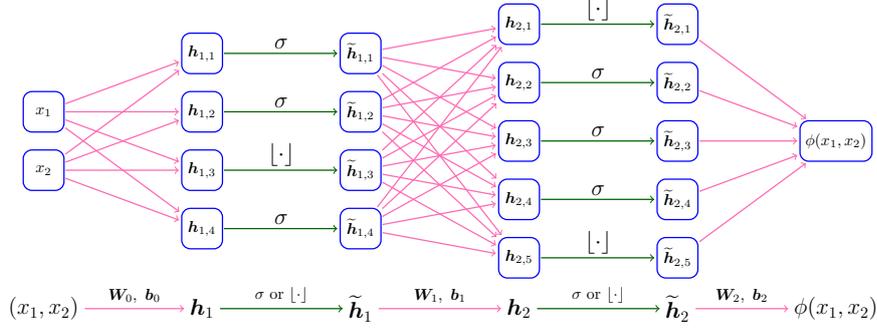


Figure 1: An example of a Floor-ReLU network with width 5 and depth 2.

72 In Theorem 1.1 below, we show by construction that Floor-ReLU networks with
73 width $\max\{d, 5N + 13\}$ and depth $64dL + 3$ can uniformly approximate a continu-
74 ous function f on $[0, 1]^d$ with a root exponential approximation rate² $\omega_f(\sqrt{d}N^{-\sqrt{L}}) +$
75 $2\omega_f(\sqrt{d})N^{-\sqrt{L}}$, where $\omega_f(\cdot)$ is the modulus of continuity defined as

$$76 \quad \omega_f(r) := \sup \{ |f(\mathbf{x}) - f(\mathbf{y})| : \|\mathbf{x} - \mathbf{y}\|_2 \leq r, \mathbf{x}, \mathbf{y} \in [0, 1]^d \}, \quad \text{for any } r \geq 0,$$

77 where $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_d^2}$ for any $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$.

78 **Theorem 1.1.** *Given any $N, L \in \mathbb{N}^+$ and an arbitrary continuous function f on $[0, 1]^d$,*
79 *there exists a function ϕ implemented by a Floor-ReLU network with width $\max\{d, 5N +$*
80 *13} and depth $64dL + 3$ such that*

$$81 \quad |\phi(\mathbf{x}) - f(\mathbf{x})| \leq \omega_f(\sqrt{d}N^{-\sqrt{L}}) + 2\omega_f(\sqrt{d})N^{-\sqrt{L}}, \quad \text{for any } \mathbf{x} \in [0, 1]^d.$$

82 With Theorem 1.1, we have an immediate corollary.

83 **Corollary 1.2.** *Given an arbitrary continuous function f on $[0, 1]^d$, there exists a func-*
84 *tion ϕ implemented by a Floor-ReLU network with width \bar{N} and depth \bar{L} such that*

$$85 \quad |\phi(\mathbf{x}) - f(\mathbf{x})| \leq \omega_f\left(\sqrt{d} \left\lfloor \frac{\bar{N}-13}{5} \right\rfloor^{-\sqrt{\left\lfloor \frac{\bar{L}-3}{64d} \right\rfloor}}\right) + 2\omega_f(\sqrt{d}) \left\lfloor \frac{\bar{N}-13}{5} \right\rfloor^{-\sqrt{\left\lfloor \frac{\bar{L}-3}{64d} \right\rfloor}},$$

86 for any $\mathbf{x} \in [0, 1]^d$ and $\bar{N}, \bar{L} \in \mathbb{N}^+$ with $\bar{N} \geq \max\{d, 18\}$ and $\bar{L} \geq 64d + 3$.

²All the exponential convergence in this paper is root exponential convergence. Nevertheless, after the introduction, for the convenience of presentation, we will omit the prefix ‘‘root’’, as in the literature.

87 In Theorem 1.1, the rate in $\omega_f(\sqrt{d}N^{-\sqrt{L}})$ implicitly depends on N and L through
 88 the modulus of continuity of f , while the rate in $2\omega_f(\sqrt{d})N^{-\sqrt{L}}$ is explicit in N and L .
 89 Simplifying the implicit approximation rate to make it explicitly depending on N and
 90 L is challenging in general. However, if f is a Hölder continuous function on $[0, 1]^d$ of
 91 order $\alpha \in (0, 1]$ with a constant λ , i.e., $f(\mathbf{x})$ satisfying

$$92 \quad |f(\mathbf{x}) - f(\mathbf{y})| \leq \lambda \|\mathbf{x} - \mathbf{y}\|_2^\alpha, \quad \text{for any } \mathbf{x}, \mathbf{y} \in [0, 1]^d, \quad (1)$$

93 then $\omega_f(r) \leq \lambda r^\alpha$ for any $r \geq 0$. Therefore, in the case of Hölder continuous functions,
 94 the approximation rate is simplified to $3\lambda d^{\alpha/2} N^{-\alpha\sqrt{L}}$ as shown in the following corol-
 95 lary. In the special case of Lipschitz continuous functions with a Lipschitz constant λ ,
 96 the approximation rate is simplified to $3\lambda\sqrt{d}N^{-\sqrt{L}}$.

97 **Corollary 1.3.** *Given any $N, L \in \mathbb{N}^+$ and a Hölder continuous function f on $[0, 1]^d$
 98 of order α with a constant λ , there exists a function ϕ implemented by a Floor-ReLU
 99 network with width $\max\{d, 5N + 13\}$ and depth $64dL + 3$ such that*

$$100 \quad |\phi(\mathbf{x}) - f(\mathbf{x})| \leq 3\lambda d^{\alpha/2} N^{-\alpha\sqrt{L}}, \quad \text{for any } \mathbf{x} \in [0, 1]^d.$$

101 First, Theorem 1.1 and Corollary 1.3 show that the approximation capacity of deep
 102 networks for continuous functions can be nearly exponentially improved by increasing
 103 the network depth, and the approximation error can be explicitly characterized in terms
 104 of the width $\mathcal{O}(N)$ and depth $\mathcal{O}(L)$. Second, this new class of networks overcomes the
 105 curse of dimensionality in the approximation power when the modulus of continuity is
 106 moderate, since the approximation order is essentially $\omega_f(\sqrt{d}N^{-\sqrt{L}})$. Finally, apply-
 107 ing piecewise constant and integer-valued functions as activation functions and integer
 108 numbers as parameters has been explored in the study of quantized neural networks
 109 (Hubara et al., 2017; Yin et al., 2019; Bengio et al., 2013) with efficient training algo-
 110 rithms for low computational complexity (Wang et al., 2018). The floor function ($\lfloor x \rfloor$) is
 111 a piecewise constant function and can be easily implemented numerically at very little
 112 cost. Hence, the evaluation of the proposed network could be efficiently implemented
 113 in practical computation. Though there might not be an existing optimization algorithm
 114 to identify an approximant with the approximation rate in this paper, Theorem 1.1 can
 115 provide an expected accuracy before a learning task and how much the current opti-
 116 mization algorithms could be improved. Designing an efficient optimization algorithm
 117 for Floor-ReLU networks will be left as future work with several possible directions
 118 discussed later.

119 We would like to remark that an increased smoothness or regularity of the target
 120 function could improve our approximation rate but at the cost of a large prefactor. For
 121 example, to attain better approximation rates for functions in $C^s([0, 1]^d)$, it is common

122 to use Taylor expansions and derivatives, which are tools that suffer from the curse
 123 of dimensionality and will result in a large prefactor like $\mathcal{O}((s+1)^d)$ that is subject
 124 to the curse of dimensionality. Furthermore, the prospective approximation rate using
 125 smoothness is not attractive. For example, the prospective approximation rate would
 126 be $\mathcal{O}(N^{-s\sqrt{L}})$, if we use Floor-ReLU networks with width $\mathcal{O}(N)$ and depth $\mathcal{O}(L)$ to
 127 approximate functions in $C^s([0,1]^d)$. However, such a rate $\mathcal{O}(N^{-s\sqrt{L}}) = \mathcal{O}(N^{-\sqrt{s^2L}})$
 128 can be attained by using Floor-ReLU networks with width $\mathcal{O}(N)$ and depth $\mathcal{O}(s^2L)$ to
 129 approximate Lipschitz continuous functions. Hence, increasing the network depth can
 130 result in the same approximation rate for Lipschitz continuous functions as the rate of
 131 smooth functions.

132 The rest of this paper is organized as follows. In Section 2, we discuss the applica-
 133 tion scope of our theory and compare related works in the literature. In Section 3, we
 134 prove Theorem 1.1 based on Proposition 3.2. Next, this basic proposition is proved in
 135 Section 4. Finally, we conclude this paper in Section 5.

136 2 Discussion

137 In this section, we will discuss the application scope of our theory in machine learning
 138 and its comparison related to existing works.

139 2.1 Application scope of our theory in machine learning

140 In supervised learning, an unknown target function $f(\mathbf{x})$ defined on a domain Ω is
 141 learned through its finitely many samples $\{(\mathbf{x}_i, f(\mathbf{x}_i))\}_{i=1}^n$. If deep networks are ap-
 142 plied in supervised learning, the following optimization problem is solved to identify a
 143 deep network $\phi(\mathbf{x}; \boldsymbol{\theta}_S)$, with $\boldsymbol{\theta}_S$ as the set of parameters, to infer $f(\mathbf{x})$ for unseen data
 144 samples \mathbf{x} :

$$145 \quad \boldsymbol{\theta}_S = \arg \min_{\boldsymbol{\theta}} R_S(\boldsymbol{\theta}) := \arg \min_{\boldsymbol{\theta}} \frac{1}{n} \sum_{\{\mathbf{x}_i\}_{i=1}^n} \ell(\phi(\mathbf{x}_i; \boldsymbol{\theta}), f(\mathbf{x}_i)) \quad (2)$$

146 with a loss function typically taken as $\ell(y, y') = \frac{1}{2}|y - y'|^2$. The inference error is usually
 147 measured by $R_{\mathcal{D}}(\boldsymbol{\theta}_S)$, where

$$148 \quad R_{\mathcal{D}}(\boldsymbol{\theta}) := \mathbb{E}_{\mathbf{x} \sim U(\Omega)} [\ell(\phi(\mathbf{x}; \boldsymbol{\theta}), f(\mathbf{x}))],$$

149 where the expectation is taken with an unknown data distribution $U(\Omega)$ over Ω .

150 Note that the best deep network to infer $f(\mathbf{x})$ is $\phi(\mathbf{x}; \boldsymbol{\theta}_{\mathcal{D}})$ with $\boldsymbol{\theta}_{\mathcal{D}}$ given by

$$151 \quad \boldsymbol{\theta}_{\mathcal{D}} = \arg \min_{\boldsymbol{\theta}} R_{\mathcal{D}}(\boldsymbol{\theta}).$$

152 The best possible inference error is $R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{D}})$. In real applications, $U(\Omega)$ is unknown
 153 and only finitely many samples from this distribution are available. Hence, the empiri-
 154 cal loss $R_{\mathcal{S}}(\boldsymbol{\theta})$ is minimized hoping to obtain $\phi(\boldsymbol{x}; \boldsymbol{\theta}_{\mathcal{S}})$, instead of minimizing the pop-
 155 ulation loss $R_{\mathcal{D}}(\boldsymbol{\theta})$ to obtain $\phi(\boldsymbol{x}; \boldsymbol{\theta}_{\mathcal{D}})$. In practice, a numerical optimization method to
 156 solve (2) may result in a numerical solution (denoted as $\boldsymbol{\theta}_{\mathcal{N}}$) that may not be a global
 157 minimizer $\boldsymbol{\theta}_{\mathcal{S}}$. Therefore, the actually learned neural network to infer $f(\boldsymbol{x})$ is $\phi(\boldsymbol{x}; \boldsymbol{\theta}_{\mathcal{N}})$
 158 and the corresponding inference error is measured by $R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{N}})$.

159 By the discussion just above, it is crucial to quantify $R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{N}})$ to see how good the
 160 learned neural network $\phi(\boldsymbol{x}; \boldsymbol{\theta}_{\mathcal{N}})$ is, since $R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{N}})$ is the expected inference error over
 161 all possible data samples. Note that

$$\begin{aligned}
 162 \quad R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{N}}) &= [R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{N}}) - R_{\mathcal{S}}(\boldsymbol{\theta}_{\mathcal{N}})] + [R_{\mathcal{S}}(\boldsymbol{\theta}_{\mathcal{N}}) - R_{\mathcal{S}}(\boldsymbol{\theta}_{\mathcal{S}})] + [R_{\mathcal{S}}(\boldsymbol{\theta}_{\mathcal{S}}) - R_{\mathcal{S}}(\boldsymbol{\theta}_{\mathcal{D}})] \\
 163 \quad &\quad + [R_{\mathcal{S}}(\boldsymbol{\theta}_{\mathcal{D}}) - R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{D}})] + R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{D}}) \\
 164 \quad &\leq R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{D}}) + [R_{\mathcal{S}}(\boldsymbol{\theta}_{\mathcal{N}}) - R_{\mathcal{S}}(\boldsymbol{\theta}_{\mathcal{S}})] \\
 165 \quad &\quad + [R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{N}}) - R_{\mathcal{S}}(\boldsymbol{\theta}_{\mathcal{N}})] + [R_{\mathcal{S}}(\boldsymbol{\theta}_{\mathcal{D}}) - R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{D}})], \tag{3}
 \end{aligned}$$

166 where the inequality comes from the fact that $[R_{\mathcal{S}}(\boldsymbol{\theta}_{\mathcal{S}}) - R_{\mathcal{S}}(\boldsymbol{\theta}_{\mathcal{D}})] \leq 0$ since $\boldsymbol{\theta}_{\mathcal{S}}$ is a
 167 global minimizer of $R_{\mathcal{S}}(\boldsymbol{\theta})$. The constructive approximation established in this paper
 168 and in the literature provides an upper bound of $R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{D}})$ in terms of the network size,
 169 e.g., in terms of the network width and depth, or in terms of the number of param-
 170 eters. The second term of (3) is bounded by the optimization error of the numerical
 171 algorithm applied to solve the empirical loss minimization problem in (2). If the nu-
 172 merical algorithm is able to find a global minimizer, the second term is equal to zero.
 173 The theoretical guarantee of the convergence of an optimization algorithm to a global
 174 minimizer $\boldsymbol{\theta}_{\mathcal{S}}$ and the characterization of the convergence belong to the optimization
 175 analysis of neural networks. The third and fourth term of (3) are usually bounded in
 176 terms of the sample size n and a certain norm of $\boldsymbol{\theta}_{\mathcal{N}}$ and $\boldsymbol{\theta}_{\mathcal{D}}$ (e.g., ℓ_1 , ℓ_2 , or the path
 177 norm), respectively. The study of the bounds for the third and fourth terms is referred
 178 to as the generalization error analysis of neural networks.

179 The approximation theory, the optimization theory, and the generalization theory
 180 form the three main theoretical aspects of deep learning with different emphases and
 181 challenges, which have motivated many separate research directions recently. Theorem
 182 1.1 and Corollary 1.3 provide an upper bound of $R_{\mathcal{D}}(\boldsymbol{\theta}_{\mathcal{D}})$. This bound only depends on
 183 the given budget of neurons and layers of Floor-ReLU networks and on the modulus of
 184 continuity of the target function f . Hence, this bound is independent of the empirical
 185 loss minimization in (2) and the optimization algorithm used to compute the numerical
 186 solution of (2). In other words, Theorem 1.1 and Corollary 1.3 quantify the approxima-
 187 tion power of Floor-ReLU networks with a given size. Designing efficient optimization
 188 algorithms and analyzing the generalization bounds for Floor-ReLU networks are two

189 other separate future directions. Although optimization algorithms and generalization
190 analysis are not our focus in this paper, in the next two paragraphs, we discuss several
191 possible research topics in these directions for our Floor-ReLU networks.

192 In this work, we have not analyzed the feasibility of optimization algorithms for the
193 Floor-ReLU network. Typically, stochastic gradient descent (SGD) is applied to solve
194 a network optimization problem. However, the Floor-ReLU network has piecewise
195 constant activation functions making standard SGD infeasible. There are two possible
196 directions to solve the optimization problem for Floor-ReLU networks: 1) gradient-free
197 optimization methods, e.g., Nelder-Mead method (Nelder and Mead, 1965), genetic al-
198 gorithm (Holland, 1992), simulated annealing (Kirkpatrick et al., 1983), particle swarm
199 optimization (Kennedy and Eberhart, 1995), and consensus-based optimization (Pinnau
200 et al., 2017; Carrillo et al., 2019); 2) applying optimization algorithms for quantized
201 networks that also have piecewise constant activation functions (Lin et al., 2019; Boo
202 et al., 2020; Bengio et al., 2013; Wang et al., 2018; Hubara et al., 2017; Yin et al., 2019).
203 It would be interesting future work to explore efficient learning algorithms based on the
204 Floor-ReLU network.

205 Generalization analysis of Floor-ReLU networks is also an interesting future di-
206 rection. Previous works have shown the generalization power of ReLU networks for
207 regression problems (Jacot et al., 2018; Cao and Gu, 2019; Chen et al., 2019b; E et al.,
208 2019; E and Wojtowytsch, 2020) and for solving partial differential equations (Berner
209 et al., 2018; Luo and Yang, 2020). Regularization strategies for ReLU networks to guar-
210 antee good generalization capacity of deep learning have been proposed in (E et al.,
211 2019; E and Wojtowytsch, 2020). It is important to investigate the generalization ca-
212 pacity of our Floor-ReLU networks. Especially, it is of great interest to see whether
213 problem-dependent regularization strategies exist to make the generalization error of
214 our Floor-ReLU networks free of the curse of dimensionality.

215 **2.2 Approximation rates in $\mathcal{O}(N)$ and $\mathcal{O}(L)$ versus $\mathcal{O}(W)$**

216 Characterizing deep network approximation in terms of the width $\mathcal{O}(N)$ ³ and depth
217 $\mathcal{O}(L)$ simultaneously is fundamental and indispensable in realistic applications, while
218 quantifying the deep network approximation based on the number of nonzero param-
219 eters W is probably only of interest in theory as far as we know. Theorem 1.1 can
220 provide practical guidance for choosing network sizes in realistic applications while
221 theories in terms of W cannot tell how large a network should be to guarantee a target
222 accuracy. The width and depth are the two most direct and amenable hyper-parameters
223 in choosing a specific network for a learning task, while the number of nonzero parame-

³For simplicity, we omit $\mathcal{O}(\cdot)$ in the following discussion.

224 ters W is hardly controlled efficiently. Theories in terms of W essentially have a single
 225 variable to control the network size in three types of structures: 1) fixing the width N
 226 and varying the depth L ; 2) fixing the depth L and changing the width N ; 3) both the
 227 width and depth are controlled by the same parameter like the target accuracy ε in a
 228 specific way (e.g., N is a polynomial of $\frac{1}{\varepsilon^d}$ and L is a polynomial of $\log(\frac{1}{\varepsilon})$). Con-
 229 sidering the non-uniqueness of structures for realizing the same W , it is impractical to
 230 develop approximation rates in terms of W covering all these structures. If one network
 231 structure has been chosen in a certain application, there might not be a known theory
 232 in terms of W to quantify the performance of this structure. Finally, in terms of full er-
 233 ror analysis of deep learning including approximation theory, optimization theory, and
 234 generalization theory as illustrated in (3), the approximation error characterization in
 235 terms of width and depth is more useful than that in terms of the number of parameters,
 236 because almost all existing optimization and generalization analysis are based on depth
 237 and width instead of the number of parameters (Jacot et al., 2018; Cao and Gu, 2019;
 238 Chen et al., 2019b; Arora et al., 2019; Allen-Zhu et al., 2019; E et al., 2019; E and
 239 Wojtowysch, 2020; Ji and Telgarsky, 2020), to the best of our knowledge. Approximi-
 240 tion results in terms of width and depth are more consistent with optimization and
 241 generalization analysis tools to obtain a full error analysis in (3).

242 Most existing approximation theories for deep neural networks so far focus on the
 243 approximation rate in the number of parameters W (Cybenko, 1989; Hornik et al.,
 244 1989; Barron, 1993; Liang and Srikant, 2016; Yarotsky, 2017; Poggio et al., 2017; E
 245 and Wang, 2018; Petersen and Voigtlaender, 2018; Chui et al., 2018; Yarotsky, 2018;
 246 Nakada and Imaizumi, 2019; Gribonval et al., 2019; Gühring et al., 2019; Chen et al.,
 247 2019a; Li et al., 2019; Suzuki, 2019; Bao et al., 2019; Opschoor et al., 2019; Yarot-
 248 sky and Zhevnerchuk, 2019; Böleskei et al., 2019; Montanelli and Du, 2019; Chen and
 249 Wu, 2019; Zhou, 2020; Montanelli and Yang, 2020; Montanelli et al., 2020). From the
 250 point of view of theoretical difficulty, controlling two variables N and L in our theory
 251 is more challenging than controlling one variable W in the literature. In terms of math-
 252 ematical logic, the characterization of deep network approximation in terms of N and
 253 L can provide an approximation rate in terms of W , while we are not aware of how to
 254 derive approximation rates in terms of arbitrary N and L given approximation rates in
 255 terms of W , since existing results in terms of W are valid for specific network sizes
 256 with width and depth as functions in W without the degree of freedom to take arbitrary
 257 values. As we have discussed in the last paragraph, existing theories essentially have a
 258 single variable to control the network size in three types of structures. Let us use the
 259 first type of structures, which includes the best-known result for a nearly optimal ap-
 260 proximation rate, $\mathcal{O}(\omega_f(W^{-2/d}))$, for continuous functions in terms of W using ReLU
 261 networks (Yarotsky, 2018) and the best-known result, $\mathcal{O}(\exp(-c_{\alpha,d}\sqrt{W}))$, for Hölder

262 continuous functions of order α using Sine-ReLU networks (Yarotsky and Zhevner-
 263 chuk, 2019), as an example to show how Theorem 1.1 in terms of N and L can be
 264 applied to show a better result in terms of W . One can apply Theorem 1.1 in a similar
 265 way to obtain other corollaries with other types of structures in terms of W . The main
 266 idea is to specify the value of N and L in Theorem 1.1 to show the desired corollary.
 267 For example, if we let the width parameter $N = 2$ and the depth parameter $L = W$ in
 268 Theorem 1.1, then the width is $\max\{d, 23\}$, the depth is $64dW + 3$, and the total num-
 269 ber of parameters is bounded by $\mathcal{O}(\max\{d^2, 23^2\}(64dW + 3)) = \mathcal{O}(W)$. Therefore,
 270 we can prove Corollary 2.1 below for the approximation capacity of our Floor-ReLU
 271 networks in terms of the total number of parameters as follows.

272 **Corollary 2.1.** *Given any $W \in \mathbb{N}^+$ and a continuous function f on $[0, 1]^d$, there exists*
 273 *a function ϕ implemented by a Floor-ReLU network with $\mathcal{O}(W)$ nonzero parameters, a*
 274 *width $\max\{d, 23\}$ and depth $64dW + 3$, such that*

$$275 \quad |\phi(\mathbf{x}) - f(\mathbf{x})| \leq \omega_f(\sqrt{d}2^{-\sqrt{W}}) + 2\omega_f(\sqrt{d})2^{-\sqrt{W}}, \quad \text{for any } \mathbf{x} \in [0, 1]^d.$$

276 Corollary 2.1 achieves root exponential convergence without the curse of dimen-
 277 sionality in terms of the number of parameters W with the help of the Floor-ReLU
 278 networks. When only ReLU networks are used, the result in (Yarotsky, 2018) suffers
 279 from the curse and does not have any kind of exponential convergence. The result in
 280 (Yarotsky and Zhevnerchuk, 2019) with Sine-ReLU networks has root exponential con-
 281 vergence but has not excluded the possibility of the curse of dimensionality as we shall
 282 discuss later. Furthermore, Corollary 2.1 works for generic continuous functions while
 283 (Yarotsky and Zhevnerchuk, 2019) only applies to Hölder continuous functions.

284 2.3 Further interpretation of our theory

285 In the interpretation of our theory, there are two more aspects that are important to
 286 discuss. The first one is whether it is possible to extend our theory to functions on
 287 a more general domain, e.g. $[-M, M]^d$ for some $M > 1$, because $M > 1$ may cause
 288 an implicit curse of dimensionality in some existing theory as we shall point out later.
 289 The second one is how bad the modulus of continuity would be since it is related to a
 290 high-dimensional function f that may lead to an implicit curse of dimensionality in our
 291 approximation rate.

292 First, Theorem 1.1 can be easily generalized to $C([-M, M]^d)$ for any $M > 0$.
 293 Let \mathcal{L} be a linear map given by $\mathcal{L}(\mathbf{x}) = 2M(\mathbf{x} - 1/2)$. By Theorem 1.1, for any
 294 $f \in C([-M, M]^d)$, there exists ϕ implemented by a Floor-ReLU network with width
 295 $\max\{d, 5N + 13\}$ and depth $64dL + 3$ such that

$$296 \quad |\phi(\mathbf{x}) - f \circ \mathcal{L}(\mathbf{x})| \leq \omega_{f \circ \mathcal{L}}(\sqrt{d}N^{-\sqrt{L}}) + 2\omega_{f \circ \mathcal{L}}(\sqrt{d})N^{-\sqrt{L}}, \quad \text{for any } \mathbf{x} \in [0, 1]^d.$$

297 It follows from $\mathbf{y} = \mathcal{L}(\mathbf{x}) \in [-M, M]^d$ and $\omega_{f \circ \mathcal{L}}(r) = \omega_f^{[-M, M]^d}(2Mr)$ for any $r \geq 0$
 298 that,⁴ for any $\mathbf{y} \in [-M, M]^d$,

$$299 \quad \left| \phi\left(\frac{\mathbf{y}+M}{2M}\right) - f(\mathbf{y}) \right| \leq \omega_f^{[-M, M]^d}(2M\sqrt{d}N^{-\sqrt{L}}) + 2\omega_f^{[-M, M]^d}(2M\sqrt{d})N^{-\sqrt{L}}. \quad (4)$$

300 Hence, the size of the function domain $[-M, M]^d$ only has a mild influence on the
 301 approximation rate of our Floor-ReLU networks. Floor-ReLU networks can still avoid
 302 the curse of dimensionality and achieve root exponential convergence for continuous
 303 functions on $[-M, M]^d$ when $M > 1$. For example, in the case of Hölder continuous
 304 functions of order α with a constant λ on $[-M, M]^d$, our approximation rate becomes
 305 $3\lambda(2M\sqrt{d}N^{-\sqrt{L}})^\alpha$.

306 Second, most interesting continuous functions in practice have a good modulus of
 307 continuity such that there is no implicit curse of dimensionality hiding in $\omega_f(\cdot)$. For
 308 example, we have discussed the case of Hölder continuous functions previously. We
 309 would like to remark that the class of Hölder continuous functions implicitly depends
 310 on d through its definition in (1), but this dependence is moderate since the ℓ^2 - norm
 311 in (1) is the square root of a sum with d terms. Let us now discuss several cases of
 312 $\omega_f(\cdot)$ when we cannot achieve exponential convergence or cannot avoid the curse of
 313 dimensionality. The first example is $\omega_f(r) = \frac{1}{\ln(1/r)}$ for all small $r > 0$, which leads to
 314 an approximation rate

$$315 \quad 3(\sqrt{L} \ln N - \frac{1}{2} \ln d)^{-1}, \quad \text{for large } N, L \in \mathbb{N}^+.$$

316 Apparently, the above approximation rate still avoids the curse of dimensionality but
 317 there is no exponential convergence, which has been canceled out by “ln” in $\omega_f(\cdot)$. The
 318 second example is $\omega_f(r) = \frac{1}{\ln^{1/d}(1/r)}$ for all small $r > 0$, which leads to an approximation
 319 rate

$$320 \quad 3(\sqrt{L} \ln N - \frac{1}{2} \ln d)^{-1/d}, \quad \text{for large } N, L \in \mathbb{N}^+.$$

321 The power $\frac{1}{d}$ further weakens the approximation rate and hence the curse of dimension-
 322 ality occurs. The last example we would like to discuss is $\omega_f(r) = r^{\alpha/d}$ for all small
 323 $r > 0$, which results in the approximation rate

$$324 \quad 3d^{\frac{\alpha}{2d}} N^{-\frac{\alpha}{d}\sqrt{L}}, \quad \text{for large } N, L \in \mathbb{N}^+,$$

325 which achieves the exponential convergence and avoids the curse of dimensionality
 326 when we use very deep networks with a fixed width. But if we fix the depth, there is no
 327 exponential convergence and the curse occurs. Though we have provided several exam-
 328 ples of immoderate $\omega_f(\cdot)$, to the best of our knowledge, we are not aware of practically
 329 useful continuous functions with $\omega_f(\cdot)$ that is immoderate.

⁴For an arbitrary set $E \subseteq \mathbb{R}^d$, $\omega_f^E(r)$ is defined via $\omega_f^E(r) := \sup \{|f(\mathbf{x}) - f(\mathbf{y})| : \|\mathbf{x} - \mathbf{y}\|_2 \leq r, \mathbf{x}, \mathbf{y} \in E\}$, for any $r \geq 0$. As defined earlier, $\omega_f(r)$ is short of $\omega_f^{[0,1]^d}(r)$.

330 2.4 Discussion on the literature

331 The neural networks constructed here achieve exponential convergence without the
332 curse of dimensionality simultaneously for a function class as general as (Hölder) con-
333 tinuous functions, while—to the best of our knowledge—most existing theories only apply
334 to functions with an intrinsic low complexity. For example, the exponential convergence
335 was studied for polynomials (Yarotsky, 2017; Montanelli et al., 2020; Lu et al., 2020),
336 smooth functions (Montanelli et al., 2020; Liang and Srikant, 2016), analytic functions
337 (E and Wang, 2018), and functions admitting a holomorphic extension to a Bernstein
338 polyellipse (Opschoor et al., 2019). For another example, no curse of dimensionality
339 occurs, or the curse is lessened for Barron spaces (Barron, 1993; E et al., 2019; E and
340 Wojtowytsch, 2020), Korobov spaces (Montanelli and Du, 2019), band-limited func-
341 tions (Chen and Wu, 2019; Montanelli et al., 2020), compositional functions (Poggio
342 et al., 2017), and smooth functions (Yarotsky and Zhevnerchuk, 2019; Lu et al., 2020;
343 Montanelli and Yang, 2020; Yang and Wang, 2020).

344 Our theory admits a neat and explicit approximation error bound. For example, our
345 approximation rate in the case of Hölder continuous functions of order α with a constant
346 λ is $3\lambda d^{\alpha/2} N^{-\alpha\sqrt{L}}$, while the prefactor of most existing theories is unknown or grows
347 exponentially in d . Our proof fully explores the advantage of the compositional struc-
348 ture and the nonlinearity of deep networks, while many existing theories were built on
349 traditional approximation tools (e.g., polynomial approximation, multiresolution anal-
350 ysis, and Monte Carlo sampling), making it challenging for existing theories to obtain
351 a neat and explicit error bound with an exponential convergence and without the curse
352 of dimensionality.

353 Let us review existing works in more detail below.

354 **Curse of dimensionality.** The curse of dimensionality is the phenomenon that ap-
355 proximating a d -dimensional function using a certain parametrization method with a
356 fixed target accuracy generally requires a large number of parameters that is exponen-
357 tial in d and this expense quickly becomes unaffordable when d is large. For example,
358 traditional finite element methods with W parameters can achieve an approximation
359 accuracy $O(W^{-1/d})$ with an explicit indicator of the curse $\frac{1}{d}$ in the power of W . If an
360 approximation rate has a constant independent of W and exponential in d , the curse still
361 occurs implicitly through this prefactor by definition. If the approximation rate has a
362 prefactor C_f depending on f , then the prefactor C_f still depends on d implicitly via f
363 and the curse implicitly occurs if C_f exponentially grows when d increases. Designing
364 a parametrization method that can overcome the curse of dimensionality is an important
365 research topic in approximation theory.

366 In (Barron, 1993) and its variants or generalization (E et al., 2019; E and Woj-
367 towytsch, 2020; Chen and Wu, 2019; Montanelli et al., 2020), d -dimensional functions

368 defined on a domain $\Omega \subseteq \mathbb{R}^d$ admitting an integral representation with an integrand as a
 369 ridge function on $\tilde{\Omega} \subseteq \mathbb{R}^d$ with a variable coefficient were considered, e.g.,

$$370 \quad f(\mathbf{x}) = \int_{\tilde{\Omega}} a(\mathbf{w})K(\mathbf{w} \cdot \mathbf{x})d\nu(\mathbf{w}), \quad (5)$$

371 where $\nu(\mathbf{w})$ is a Lebesgue measure in \mathbf{w} . $f(\mathbf{x})$ can be reformulated into the expectation
 372 of a high-dimensional random function when \mathbf{w} is treated as a random variable. Then
 373 $f(\mathbf{x})$ can be approximated by the average of W samples of the integrand in the same
 374 spirit of the law of large numbers with an approximation error essentially bounded
 375 by $\frac{C_f\sqrt{\mu(\Omega)}}{\sqrt{W}}$ measured in $L^2(\Omega, \mu)$ (Equation (6) of (Barron, 1993)), where $\mathcal{O}(W)$ is
 376 the total number of parameters in the network, C_f is a d -dimensional integral with an
 377 integrand related to f , and $\mu(\Omega)$ is the Lebesgue measure of Ω . As pointed out in
 378 (Barron, 1993) right after Equation (6), if Ω is not a unit domain in \mathbb{R}^d , $\mu(\Omega)$ would
 379 be exponential in d ; at the beginning of Page 932 of (Barron, 1993), it was remarked
 380 that C_f can often be exponentially large in d and standard smoothness properties of f
 381 alone are not enough to remove the exponential dependence of C_f on d , though there is
 382 a large number of examples for which C_f is only moderately large. Therefore, the curse
 383 of dimensionality occurs unless C_f and $\mu(\Omega)$ are not exponential in d . It was observed
 384 that if the error is measured in the sense of mean squared error in machine learning,
 385 which is the square of the $L^2(\Omega, \mu)$ error averaged over $\mu(\Omega)$ resulting in $\frac{C_f^2}{W}$, then the
 386 mean squared error has no curse of dimensionality as long as C_f is not exponential in d
 387 (Barron, 1993; E et al., 2019; E and Wojtowytsch, 2020).

388 In (Montanelli and Du, 2019), d -dimensional functions in the Korobov space are
 389 approximated by the linear combination of basis functions of a sparse grid, each of
 390 which is approximated by a ReLU network. Though the curse of dimensionality has
 391 been lessened, target functions have to be sufficiently smooth and the approximation
 392 error still contains a factor that is exponential in d , i.e., the curse still occurs. Other
 393 works in (Yarotsky, 2017; Yarotsky and Zhevnerchuk, 2019; Lu et al., 2020; Yang and
 394 Wang, 2020) study the advantage of smoothness in the network approximation. Polyno-
 395 mials are applied to approximate smooth functions and ReLU networks are constructed
 396 to approximate polynomials. The application of smoothness can lessen the curse of di-
 397 mensionality in the approximation rates in terms of network sizes but also results in a
 398 prefactor that is exponentially large in the dimension, which means that the curse still
 399 occurs implicitly.

400 The Kolmogorov-Arnold superposition theorem (KST) (Kolmogorov, 1956; Arnold,
 401 1957; Kolmogorov, 1957) has also inspired a research direction of network approxima-
 402 tion (Kůrková, 1992; Maiorov and Pinkus, 1999; Igel'nik and Parikh, 2003; Montanelli
 403 and Yang, 2020) for continuous functions. (Kůrková, 1992) provided a quantitative ap-
 404 proximation rate of networks with two hidden layers, but the number of neurons scales

405 exponentially in the dimension and the curse occurs. (Maierov and Pinkus, 1999) re-
 406 laxes the exact representation in KST to an approximation in a form of two-hidden-
 407 layer neural networks with a maximum width $6d + 3$ and a single activation function.
 408 This powerful activation function is very complex as described by its authors and its
 409 numerical evaluation was not available until a more concrete algorithm was recently
 410 proposed in (Guliyev and Ismailov, 2018). Note that there is no available numerical
 411 algorithm in (Maierov and Pinkus, 1999; Guliyev and Ismailov, 2018) to compute the
 412 whole networks proposed therein. The difficulty is due to the fact that the construction
 413 of these networks relies on the outer univariate continuous function of the KST. Though
 414 the existence of these outer functions can be shown by construction via a complicated
 415 iterative procedure in (Braun and Griebel, 2009), there is no existing numerical algo-
 416 rithm to evaluate them for a given target function yet, even though computation with
 417 an arbitrary precision is assumed to be available. Therefore, the networks considered
 418 in (Maierov and Pinkus, 1999; Guliyev and Ismailov, 2018) are similar to the original
 419 representation in KST in the sense that their existence is proved without an explicit way
 420 or numerical algorithm to construct them. (Igel'nik and Parikh, 2003) and (Montanelli
 421 and Yang, 2020) apply cubic-splines and piecewise linear functions to approximate the
 422 inner and outer functions of KST, resulting in cubic-spline and ReLU networks to ap-
 423 proximate continuous functions on $[0, 1]^d$. Due to the pathological outer functions of
 424 KST, the approximation bounds still suffer from the curse of dimensionality unless tar-
 425 get functions are restricted to a small class of functions with simple outer functions in
 426 the KST.

427 Recently in (Yarotsky and Zhevnerchuk, 2019), Sine-ReLU networks have been
 428 applied to approximate Hölder continuous functions of order α on $[0, 1]^d$ with an ap-
 429 proximation accuracy $\varepsilon = \exp(-c_{\alpha,d}W^{1/2})$, where W is the number of parameters in the
 430 network and $c_{\alpha,d}$ is a positive constant depending on α and d only. Whether or not $c_{\alpha,d}$
 431 exponentially depends on d determines whether or not the curse of dimensionality ex-
 432 ists for the Sine-ReLU networks, which is not answered in (Yarotsky and Zhevnerchuk,
 433 2019) and is still an open question.

434 Finally, we would like to discuss the curse of dimensionality in terms of the con-
 435 tinuity of the weight selection as a map $\Sigma : C([0, 1]^d) \rightarrow \mathbb{R}^W$. For a fixed network
 436 architecture with a fixed number of parameters W , let $g : \mathbb{R}^W \rightarrow C([0, 1]^d)$ be the map
 437 of realizing a DNN from a given set of parameters in \mathbb{R}^W to a function in $C([0, 1]^d)$.
 438 Suppose that there is a continuous map Σ from the unit ball of Sobolev space with
 439 smoothness s , denoted as $F_{s,d}$, to \mathbb{R}^W such that $\|f - g(\Sigma(f))\|_{L^\infty} \leq \varepsilon$ for all $f \in F_{s,d}$.
 440 Then $W \geq c\varepsilon^{-d/s}$ with some constant c depending only on s . This conclusion is given
 441 in Theorem 3 of (Yarotsky, 2017), which is a corollary of Theorem 4.2 of (Devore,
 442 1989) in a more general form. Intuitively, this conclusion means that any constructive

443 approximation of ReLU FNNs to approximate $C([0, 1]^d)$ cannot enjoy a continuous
 444 weight selection property if the approximation rate is better than $c\varepsilon^{-d/s}$, i.e., the curse
 445 of dimensionality must occur for constructive approximation for ReLU FNNs with a
 446 continuous weight selection. Theorem 4.2 of (Devore, 1989) can also lead to a new
 447 corollary with a weight selection map $\Sigma : K_{s,d} \rightarrow \mathbb{R}^W$ (e.g., the constructive approxi-
 448 mation of Floor-ReLU networks) and $g : \mathbb{R}^W \rightarrow L^\infty([0, 1]^d)$ (e.g., the realization map
 449 of Floor-ReLU networks), where $K_{s,d}$ is the unit ball of $C^s([0, 1]^d)$ with the Sobolev
 450 norm $W^{s,\infty}([0, 1]^d)$. Then this new corollary implies that the constructive approxima-
 451 tion in this paper cannot enjoy continuous weight selection. However, Theorem 4.2
 452 of (Devore, 1989) is essentially a min-max criterion to evaluate weight selection maps
 453 maintaining continuity: the approximation error obtained by minimizing over all con-
 454 tinuous selection Σ and network realization g and maximizing over all target functions
 455 is bounded below by $\mathcal{O}(W^{-s/d})$. In the worst scenario, a continuous weight selec-
 456 tion cannot enjoy an approximation rate beating the curse of dimensionality. However,
 457 Theorem 4.2 of (Devore, 1989) has not excluded the possibility that most continuous
 458 functions of interest in practice may still enjoy a continuous weight selection without
 459 the curse of dimensionality.

460 **Exponential convergence.** Exponential convergence is referred to as the situation
 461 that the approximation error exponentially decays to zero when the number of param-
 462 eters increases. Designing approximation tools with an exponential convergence is an-
 463 other important topic in approximation theory. In the literature of deep network approx-
 464 imation, when the number of network parameters W is a polynomial of $\mathcal{O}(\log(\frac{1}{\varepsilon}))$, the
 465 terminology “exponential convergence” was also used (E and Wang, 2018; Yarotsky and
 466 Zhevnerchuk, 2019; Opschoor et al., 2019). The exponential convergence in this paper
 467 is root-exponential as in (Yarotsky and Zhevnerchuk, 2019), i.e., $W = \mathcal{O}(\log^2(\frac{1}{\varepsilon}))$. The
 468 exponential convergence in other works is worse than root-exponential.

469 In most cases, the approximation power to achieve exponential approximation rates
 470 in existing works comes from traditional tools for approximating a small class of func-
 471 tions instead of taking advantage of the network structure itself. In (E and Wang, 2018;
 472 Opschoor et al., 2019), highly smooth functions are first approximated by the linear
 473 combination of special polynomials with high degrees (e.g., Chebyshev polynomials,
 474 Legendre polynomials) with an exponential approximation rate, i.e., to achieve an ε -
 475 accuracy, a linear combination of only $\mathcal{O}(p(\log(\frac{1}{\varepsilon})))$ polynomials is required, where p
 476 is a polynomial with a degree that may depend on the dimension d . Then each poly-
 477 nomial is approximated by a ReLU network with $\mathcal{O}(\log(\frac{1}{\varepsilon}))$ parameters. Finally, all
 478 ReLU networks are assembled to form a large network approximating the target func-
 479 tion with an exponential approximation rate. As far as we know, the only existing work
 480 that achieves exponential convergence without taking advantage of special polynomials

481 and smoothness is the Sine-ReLU network in (Yarotsky and Zhevnerchuk, 2019), which
 482 has been mentioned in the paragraph just above. We would like to emphasize that the
 483 result in our paper applies for generic continuous functions including, but not limited
 484 to, the Hölder continuous functions considered in (Yarotsky and Zhevnerchuk, 2019).

485 3 Approximation of continuous functions

486 In this section, we first introduce basic notations in this paper in Section 3.1. Then we
 487 prove Theorem 1.1 based on Proposition 3.2, which will be proved in Section 4.

488 3.1 Notations

489 The main notations of this paper are listed as follows.

490 • Vectors and matrices are denoted in a bold font. Standard vectorization is adopted
 491 in the matrix and vector computation. For example, adding a scalar and a vector
 492 means adding the scalar to each entry of the vector.

493 • Let \mathbb{N}^+ denote the set containing all positive integers, i.e., $\mathbb{N}^+ = \{1, 2, 3, \dots\}$.

494 • Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ denote the rectified linear unit (ReLU), i.e. $\sigma(x) = \max\{0, x\}$.

495 With a slight abuse of notation, we define $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ as $\sigma(\mathbf{x}) = \begin{bmatrix} \max\{0, x_1\} \\ \vdots \\ \max\{0, x_d\} \end{bmatrix}$

496 for any $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$.

497 • The floor function (Floor) is defined as $\lfloor x \rfloor := \max\{n : n \leq x, n \in \mathbb{Z}\}$ for any
 498 $x \in \mathbb{R}$.

499 • For $\theta \in [0, 1)$, suppose its binary representation is $\theta = \sum_{\ell=1}^{\infty} \theta_{\ell} 2^{-\ell}$ with $\theta_{\ell} \in \{0, 1\}$,
 500 we introduce a special notation $\text{bin}0.\theta_1\theta_2\cdots\theta_L$ to denote the L -term binary repre-
 501 sentation of θ , i.e., $\text{bin}0.\theta_1\theta_2\cdots\theta_L := \sum_{\ell=1}^L \theta_{\ell} 2^{-\ell}$.

502 • The expression “a network with width N and depth L ” means

503 – The maximum width of this network for all **hidden** layers is no more than
 504 N .

505 – The number of **hidden** layers of this network is no more than L .

506 **3.2 Proof of Theorem 1.1**

507 Theorem 1.1 is an immediate consequence of Theorem 3.1 below.

508 **Theorem 3.1.** *Given any $N, L \in \mathbb{N}^+$ and an arbitrary continuous function f on $[0, 1]^d$,*
 509 *there exists a function ϕ implemented by a Floor-ReLU network with width $\max\{d, 2N^2 +$
 510 $5N\}$ and depth $7dL^2 + 3$ such that*

$$511 \quad |\phi(\mathbf{x}) - f(\mathbf{x})| \leq \omega_f(\sqrt{d}N^{-L}) + 2\omega_f(\sqrt{d})2^{-NL}, \quad \text{for any } \mathbf{x} \in [0, 1]^d.$$

512 This theorem will be proved later in this section. Now let us prove Theorem 1.1
 513 based on Theorem 3.1.

514 *Proof of Theorem 1.1.* Given any $N, L \in \mathbb{N}^+$, there exist $\tilde{N}, \tilde{L} \in \mathbb{N}^+$ with $\tilde{N} \geq 2$ and
 515 $\tilde{L} \geq 3$ such that

$$516 \quad (\tilde{N} - 1)^2 \leq N < \tilde{N}^2 \quad \text{and} \quad (\tilde{L} - 1)^2 \leq 4L < \tilde{L}^2.$$

517 By Theorem 3.1, there exists a function ϕ implemented by a Floor-ReLU network with
 518 width $\max\{d, 2\tilde{N}^2 + 5\tilde{N}\}$ and depth $7d\tilde{L}^2 + 3$ such that

$$519 \quad |\phi(\mathbf{x}) - f(\mathbf{x})| \leq \omega_f(\sqrt{d}\tilde{N}^{-\tilde{L}}) + 2\omega_f(\sqrt{d})2^{-\tilde{N}\tilde{L}}, \quad \text{for any } \mathbf{x} \in [0, 1]^d.$$

520 Note that

$$521 \quad 2^{-\tilde{N}\tilde{L}} \leq \tilde{N}^{-\tilde{L}} = (\tilde{N}^2)^{-\frac{1}{2}\sqrt{\tilde{L}^2}} \leq N^{-\frac{1}{2}\sqrt{4L}} \leq N^{-\sqrt{L}}.$$

522 Then we have

$$523 \quad |\phi(\mathbf{x}) - f(\mathbf{x})| \leq \omega_f(\sqrt{d}N^{-\sqrt{L}}) + 2\omega_f(\sqrt{d})N^{-\sqrt{L}}, \quad \text{for any } \mathbf{x} \in [0, 1]^d.$$

524 For $\tilde{N}, \tilde{L} \in \mathbb{N}^+$ with $\tilde{N} \geq 2$ and $\tilde{L} \geq 3$, we have

$$525 \quad 2\tilde{N}^2 + 5\tilde{N} \leq 5(\tilde{N} - 1)^2 + 13 \leq 5N + 13 \quad \text{and} \quad 7\tilde{L}^2 \leq 16(\tilde{L} - 1)^2 \leq 64L.$$

526 Therefore, ϕ can be computed by a Floor-ReLU network with width $\max\{d, 2\tilde{N}^2 +$
 527 $5\tilde{N}\} \leq \max\{d, 5N + 13\}$ and depth $7d\tilde{L}^2 + 3 \leq 64dL + 3$, as desired. So we finish the
 528 proof. \square

529 To prove Theorem 3.1, we first present the proof sketch. Put briefly, we construct
 530 piecewise constant functions implemented by Floor-ReLU networks to approximate
 531 continuous functions. There are four key steps in our construction.

- 532 1. Normalize f as \tilde{f} satisfying $\tilde{f}(\mathbf{x}) \in [0, 1]$ for any $\mathbf{x} \in [0, 1]^d$, divide $[0, 1]^d$ into a
 533 set of non-overlapping cubes $\{Q_\beta\}_{\beta \in \{0, 1, \dots, K-1\}^d}$, and denote \mathbf{x}_β as the vertex of
 534 Q_β with minimum $\|\cdot\|_1$ norm, where K is an integer determined later. See Figure
 535 2 for the illustrations of Q_β and \mathbf{x}_β .

- 536 2. Construct a Floor-ReLU sub-network to implement a vector-valued function $\Phi_1 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ projecting the whole cube Q_β to the index β for each $\beta \in \{0, 1, \dots, K - 1\}^d$, i.e., $\Phi_1(\mathbf{x}) = \beta$ for all $\mathbf{x} \in Q_\beta$.
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538
- 539 3. Construct a Floor-ReLU sub-network to implement a function $\phi_2 : \mathbb{R}^d \rightarrow \mathbb{R}$ mapping $\beta \in \{0, 1, \dots, K - 1\}^d$ approximately to $\tilde{f}(\mathbf{x}_\beta)$ for each β , i.e., $\phi_2(\beta) \approx \tilde{f}(\mathbf{x}_\beta)$. Then $\phi_2 \circ \Phi_1(\mathbf{x}) = \phi_2(\beta) \approx \tilde{f}(\mathbf{x}_\beta)$ for any $\mathbf{x} \in Q_\beta$ and each $\beta \in \{0, 1, \dots, K - 1\}^d$, implying $\tilde{\phi} := \phi_2 \circ \Phi_1$ approximates \tilde{f} within an error $\mathcal{O}(\omega_f(1/K))$ on $[0, 1]^d$.
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- 544 4. Re-scale and shift $\tilde{\phi}$ to obtain the desired function ϕ approximating f well and determine the final Floor-ReLU network to implement ϕ .
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546 It is not difficult to construct Floor-ReLU networks with the desired width and depth to implement Φ_1 . The most technical part is the construction of a Floor-ReLU network with the desired width and depth computing ϕ_2 , which needs the following proposition based on the ‘‘bit extraction’’ technique introduced in (Bartlett et al., 1998; Harvey et al., 2017).
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551 **Proposition 3.2.** *Given any $N, L \in \mathbb{N}^+$ and arbitrary $\theta_m \in \{0, 1\}$ for $m = 1, 2, \dots, N^L$, there exists a function ϕ computed by a Floor-ReLU network with width $2N + 2$ and depth $7L - 2$ such that*
552
553

$$554 \quad \phi(m) = \theta_m, \quad \text{for } m = 1, 2, \dots, N^L.$$

555 The proof of this proposition is presented in Section 4. By this proposition and the definition of VC-dimension (e.g., see (Harvey et al., 2017)), it is easy to prove that the VC-dimension of Floor-ReLU networks with a constant width and depth $\mathcal{O}(L)$ has a lower bound 2^L . Such a lower bound is much larger than $\mathcal{O}(L^2)$, which is a VC-dimension upper bound of ReLU networks with the same width and depth due to Theorem 8 of (Harvey et al., 2017). This means Floor-ReLU networks are much more powerful than ReLU networks from the perspective of VC-dimension.
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562 Based on the proof sketch stated just above, we are ready to give the detailed proof of Theorem 3.1 following similar ideas as in our previous work (Shen et al., 2019a; Shen et al., 2019b; Lu et al., 2020). The main idea of our proof is to reduce high-dimensional approximation to one-dimensional approximation via a projection. The idea of projection was probably first used in well-established theories, e.g., KST (Kolmogorov superposition theorem) mentioned in Section 2, where the approximant to high-dimensional functions is constructed by: first, projecting high-dimensional data points to one-dimensional data points; second, construct one-dimensional approximants. There has been extensive research based on this idea, e.g., references related
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571 to KST summarized in Section 2, our previous works (Shen et al., 2019a; Shen et al.,
572 2019b; Lu et al., 2020), and (Yarotsky and Zhevnerchuk, 2019). The key to a suc-
573 cessful approximant is to construct one-dimensional approximants to deal with a large
574 number of one-dimensional data points; in fact, the number of points is exponential in
575 the dimension d .

576 *Proof of Theorem 3.1.* The proof consists of four steps.

577 **Step 1:** Set up.

578 Assume f is not a constant function since it is a trivial case. Then $\omega_f(r) > 0$ for any
579 $r > 0$. Clearly, $|f(\mathbf{x}) - f(\mathbf{0})| \leq \omega_f(\sqrt{d})$ for any $\mathbf{x} \in [0, 1]^d$. Define

$$580 \quad \tilde{f} := (f - f(\mathbf{0}) + \omega_f(\sqrt{d})) / (2\omega_f(\sqrt{d})). \quad (6)$$

581 It follows that $\tilde{f}(\mathbf{x}) \in [0, 1]$ for any $\mathbf{x} \in [0, 1]^d$.

582 Set $K = N^L$, $E_{K-1} = [\frac{K-1}{K}, 1]$, and $E_k = [\frac{k}{K}, \frac{k+1}{K})$ for $k = 0, 1, \dots, K-2$. Define
583 $\mathbf{x}_\beta := \beta/K$ and

$$584 \quad Q_\beta := \left\{ \mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_j \in E_{\beta_j} \text{ for } j = 1, 2, \dots, d \right\},$$

585 for any $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \{0, 1, \dots, K-1\}^d$. See Figure 2 for the examples of Q_β and
586 \mathbf{x}_β for $\beta \in \{0, 1, \dots, K-1\}^d$ with $K = 4$ and $d = 1, 2$.

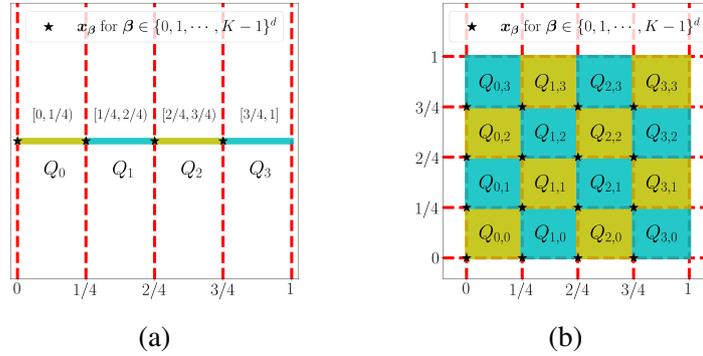


Figure 2: Illustrations of Q_β and \mathbf{x}_β for $\beta \in \{0, 1, \dots, K-1\}^d$. (a) $K = 4, d = 1$. (b) $K = 4, d = 2$.

587 **Step 2:** Construct Φ_1 mapping $\mathbf{x} \in Q_\beta$ to β .

588 Define a step function ϕ_1 as

$$589 \quad \phi_1(x) := \lfloor -\sigma(-Kx + K - 1) + K - 1 \rfloor, \quad \text{for any } x \in \mathbb{R}.^5$$

590 See Figure 3 for an example of ϕ_1 when $K = 4$. It follows from the definition of ϕ_1 that

$$591 \quad \phi_1(x) = k, \quad \text{if } x \in E_k, \text{ for } k = 0, 1, \dots, K-1.$$

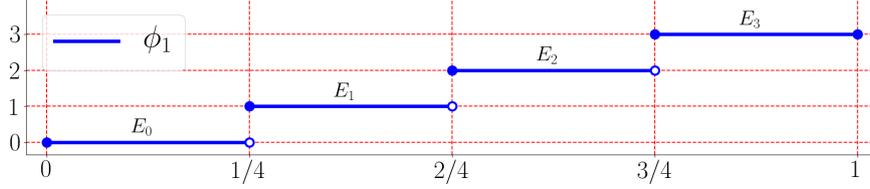


Figure 3: An illustration of ϕ_1 on $[0, 1]$ for the case $K = 4$.

592 Define

$$593 \quad \Phi_1(\mathbf{x}) := (\phi_1(x_1), \phi_1(x_2), \dots, \phi_1(x_d)), \quad \text{for any } \mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d.$$

594 Clearly, we have, for $\mathbf{x} \in Q_\beta$ and $\beta \in \{0, 1, \dots, K-1\}^d$,

$$595 \quad \Phi_1(\mathbf{x}) = (\phi_1(x_1), \phi_1(x_2), \dots, \phi_1(x_d)) = (\beta_1, \beta_2, \dots, \beta_d) = \beta.$$

596 **Step 3:** Construct ϕ_2 mapping $\beta \in \{0, 1, \dots, K-1\}^d$ approximately to $\tilde{f}(\mathbf{x}_\beta)$.

597 Using the idea of K -ary representation, we define a linear function ψ_1 via

$$598 \quad \psi_1(\mathbf{x}) := 1 + \sum_{j=1}^d x_j K^{j-1}, \quad \text{for any } \mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d.$$

599 Then ψ_1 is a bijection from $\{0, 1, \dots, K-1\}^d$ to $\{1, 2, \dots, K^d\}$.

600 Given any $i \in \{1, 2, \dots, K^d\}$, there exists a unique $\beta \in \{0, 1, \dots, K-1\}^d$ such that
601 $i = \psi_1(\beta)$. Then define

$$602 \quad \xi_i := \tilde{f}(\mathbf{x}_\beta) \in [0, 1], \quad \text{for } i = \psi_1(\beta) \text{ and } \beta \in \{0, 1, \dots, K-1\}^d,$$

603 where \tilde{f} is the normalization of f defined in Equation (6). It follows that there exists
604 $\xi_{i,j} \in \{0, 1\}$ for $j = 1, 2, \dots, NL$ such that

$$605 \quad |\xi_i - \text{bin}_{0.\xi_{i,1}\xi_{i,2}\dots\xi_{i,NL}}| \leq 2^{-NL}, \quad \text{for } i = 1, 2, \dots, K^d.$$

606 By $K^d = (NL)^d = N^{dL}$ and Proposition 3.2, there exists a function $\psi_{2,j}$ implemented
607 by a Floor-ReLU network with width $2N+2$ and depth $7dL-2$, for each $j = 1, 2, \dots, NL$,
608 such that

$$609 \quad \psi_{2,j}(i) = \xi_{i,j}, \quad \text{for } i = 1, 2, \dots, K^d.$$

610 Define

$$611 \quad \psi_2 := \sum_{j=1}^{NL} 2^{-j} \psi_{2,j} \quad \text{and} \quad \phi_2 := \psi_2 \circ \psi_1.$$

⁵If we just define $\phi_1(x) = \lfloor Kx \rfloor$, then $\phi_1(1) = K \neq K-1$ even though $1 \in E_{K-1}$.

612 Then, for $i = \psi_1(\boldsymbol{\beta})$ and $\boldsymbol{\beta} \in \{0, 1, \dots, K-1\}^d$, we have

$$\begin{aligned}
613 \quad |\tilde{f}(\mathbf{x}_\beta) - \phi_2(\boldsymbol{\beta})| &= |\tilde{f}(\mathbf{x}_\beta) - \psi_2(\psi_1(\boldsymbol{\beta}))| = |\xi_i - \psi_2(i)| = \left| \xi_i - \sum_{j=1}^{NL} 2^{-j} \psi_{2,j}(i) \right| \\
&= |\xi_i - \text{bin}_{0.\xi_{i,1}\xi_{i,2}\dots\xi_{i,NL}}| \leq 2^{-NL}.
\end{aligned} \tag{7}$$

614 **Step 4:** Determine the final network to implement the desired function ϕ .

615 Define $\tilde{\phi} := \phi_2 \circ \Phi_1$, i.e., for any $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$,

$$616 \quad \tilde{\phi}(\mathbf{x}) = \phi_2 \circ \Phi_1(\mathbf{x}) = \phi_2(\phi_1(x_1), \phi_1(x_2), \dots, \phi_1(x_d)).$$

617 Note that $|\mathbf{x} - \mathbf{x}_\beta| \leq \frac{\sqrt{d}}{K}$ for any $\mathbf{x} \in Q_\beta$ and $\boldsymbol{\beta} \in \{0, 1, \dots, K-1\}^d$. Then we have,
618 for any $\mathbf{x} \in Q_\beta$ and $\boldsymbol{\beta} \in \{0, 1, \dots, K-1\}^d$,

$$\begin{aligned}
|\tilde{f}(\mathbf{x}) - \tilde{\phi}(\mathbf{x})| &\leq |\tilde{f}(\mathbf{x}) - \tilde{f}(\mathbf{x}_\beta)| + |\tilde{f}(\mathbf{x}_\beta) - \tilde{\phi}(\mathbf{x})| \\
619 \quad &\leq \omega_{\tilde{f}}\left(\frac{\sqrt{d}}{K}\right) + |\tilde{f}(\mathbf{x}_\beta) - \phi_2(\Phi_1(\mathbf{x}))| \\
&\leq \omega_{\tilde{f}}\left(\frac{\sqrt{d}}{K}\right) + |\tilde{f}(\mathbf{x}_\beta) - \phi_2(\boldsymbol{\beta})| \leq \omega_{\tilde{f}}\left(\frac{\sqrt{d}}{K}\right) + 2^{-NL},
\end{aligned}$$

620 where the last inequality comes from Equation (7).

621 Note that $\mathbf{x} \in Q_\beta$ and $\boldsymbol{\beta} \in \{0, 1, \dots, K-1\}^d$ are arbitrary. Since $[0, 1]^d = \bigcup_{\boldsymbol{\beta} \in \{0, 1, \dots, K-1\}^d} Q_\beta$,
622 we have

$$623 \quad |\tilde{f}(\mathbf{x}) - \tilde{\phi}(\mathbf{x})| \leq \omega_{\tilde{f}}\left(\frac{\sqrt{d}}{K}\right) + 2^{-NL}, \quad \text{for any } \mathbf{x} \in [0, 1]^d.$$

624 Define

$$625 \quad \phi := 2\omega_f(\sqrt{d})\tilde{\phi} + f(\mathbf{0}) - \omega_f(\sqrt{d}).$$

626 By $K = N^L$ and $\omega_f(r) = 2\omega_f(\sqrt{d}) \cdot \omega_{\tilde{f}}(r)$ for any $r \geq 0$, we have, for any $\mathbf{x} \in [0, 1]^d$,

$$\begin{aligned}
|f(\mathbf{x}) - \phi(\mathbf{x})| &= 2\omega_f(\sqrt{d})|\tilde{f}(\mathbf{x}) - \tilde{\phi}(\mathbf{x})| \leq 2\omega_f(\sqrt{d})\left(\omega_{\tilde{f}}\left(\frac{\sqrt{d}}{K}\right) + 2^{-NL}\right) \\
627 \quad &\leq \omega_f\left(\frac{\sqrt{d}}{K}\right) + 2\omega_f(\sqrt{d})2^{-NL} \\
&\leq \omega_f(\sqrt{d}N^{-L}) + 2\omega_f(\sqrt{d})2^{-NL}.
\end{aligned}$$

628 It remains to determine the width and depth of the Floor-ReLU network implement-
629 ing ϕ . Clearly, ϕ_2 can be implemented by the architecture in Figure 4.

630 As we can see from Figure 4, ϕ_2 can be implemented by a Floor-ReLU network with
631 width $N(2N+2+3) = 2N^2+5N$ and depth $L(7dL-2+1)+2 = L(7dL-1)+2$. With the
632 network architecture implementing ϕ_2 in hand, $\tilde{\phi}$ can be implemented by the network
633 architecture shown in Figure 5. Note that ϕ is defined via re-scaling and shifting $\tilde{\phi}$. As
634 shown in Figure 5, ϕ and $\tilde{\phi}$ can be implemented by a Floor-ReLU network with width
635 $\max\{d, 2N^2+5N\}$ and depth $1+1+L(7dL-1)+2 \leq 7dL^2+3$. So we finish the proof.

636 \square

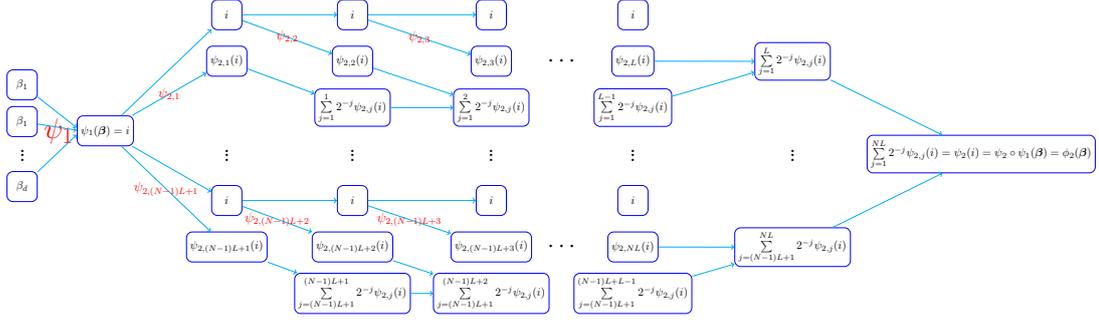


Figure 4: An illustration of the desired network architecture implementing $\phi_2 = \psi_2 \circ \psi_1$ for any input $\beta \in \{0, 1, \dots, K-1\}^d$, where $i = \psi_1(\beta)$.

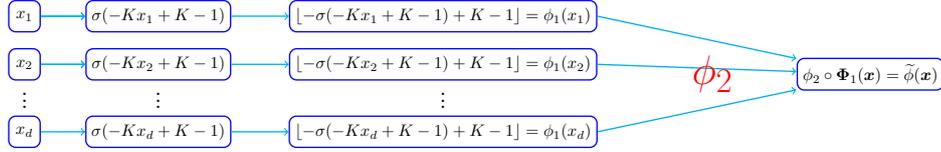


Figure 5: An illustration of the network architecture implementing $\tilde{\phi} = \phi_2 \circ \Phi_1$.

637 4 Proof of Proposition 3.2

638 The proof of Proposition 3.2 mainly relies on the “bit extraction” technique. As we shall
 639 see later, our key idea is to apply the Floor activation function to make “bit extraction”
 640 more powerful to reduce network sizes. In particular, Floor-ReLU networks can extract
 641 much more bits than ReLU networks with the same network size.

642 Let us first establish a basic lemma to extract $1/N$ of the total bits of a binary
 643 number; the result is again stored in a binary number.

644 **Lemma 4.1.** *Given any $J, N \in \mathbb{N}^+$, there exists a function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ that can be*
 645 *implemented by a Floor-ReLU network with width $2N$ and depth 4 such that, for any*
 646 *$\theta_j \in \{0, 1\}$, $j = 1, \dots, NJ$, we have*

$$647 \quad \phi(\text{bin}0.\theta_1 \cdots \theta_{NJ}, n) = \text{bin}0.\theta_{(n-1)J+1} \cdots \theta_{nJ}, \quad \text{for } n = 1, 2, \dots, N.$$

648 *Proof.* Given any $\theta_j \in \{0, 1\}$ for $j = 1, \dots, NJ$, denote

$$649 \quad s = \text{bin}0.\theta_1 \cdots \theta_{NJ} \quad \text{and} \quad s_n = \text{bin}0.\theta_{(n-1)J+1} \cdots \theta_{nJ}, \quad \text{for } n = 1, 2, \dots, N.$$

650 Then our goal is to construct a function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ computed by a Floor-ReLU
 651 network with the desired width and depth that satisfies

$$652 \quad \phi(s, n) = s_n, \quad \text{for } n = 1, 2, \dots, N.$$

653 Based on the properties of the binary representation, it is easy to check that

$$654 \quad s_n = \lfloor 2^{nJ} s \rfloor / 2^J - \lfloor 2^{(n-1)J} s \rfloor, \quad \text{for } n = 1, 2, \dots, N. \quad (8)$$

655 Even with the above formulas to generate s_1, s_2, \dots, s_N , it is still technical to construct
 656 a network outputting s_n for a given index $n \in \{1, 2, \dots, N\}$.

657 Set $\delta = 2^{-J}$ and define g (see Figure 6) as

658
$$g(x) := \sigma\left(\sigma(x) - \sigma\left(\frac{x+\delta-1}{\delta}\right)\right), \quad \text{where } \sigma(x) = \max\{0, x\}.$$

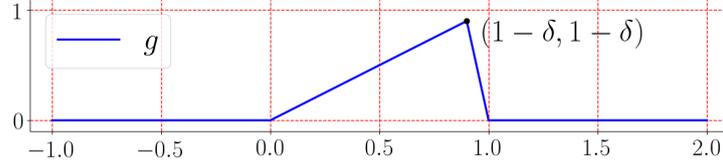


Figure 6: An illustration of $g(x) = \sigma\left(\sigma(x) - \sigma\left(\frac{x+\delta-1}{\delta}\right)\right)$, where $\sigma(x) = \max\{0, x\}$ is the ReLU activation function.

659 Since $s_n \in [0, 1 - \delta]$ for $n = 1, 2, \dots, N$, we have

660
$$s_n = \sum_{k=1}^N g(s_k + k - n), \quad \text{for } n = 1, 2, \dots, N. \quad (9)$$

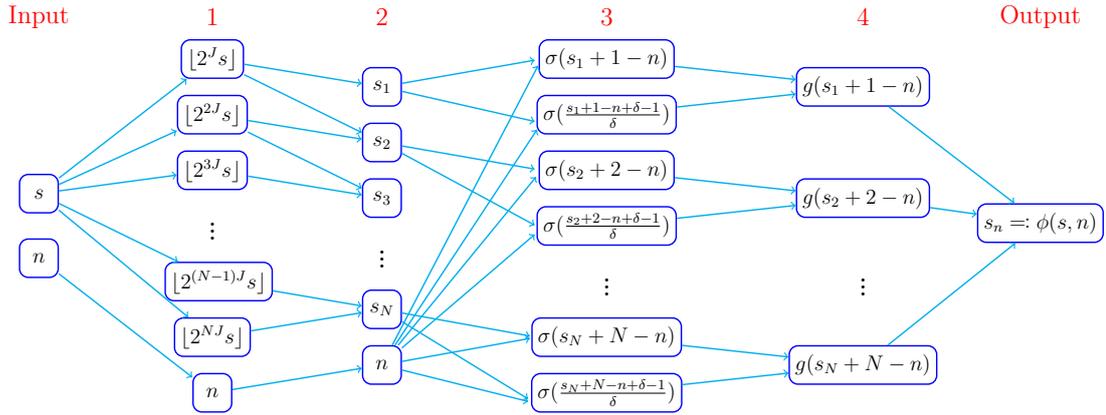


Figure 7: An illustration of the desired network architecture implementing ϕ based on Equation (8) and (9). We omit some ReLU (σ) activation functions when inputs are obviously non-negative. All parameters in this network are essentially determined by Equation (8) and (9), which are valid no matter what $\theta_1, \dots, \theta_{NJ} \in \{0, 1\}$ are. Thus, the desired function ϕ implemented by this network is independent of $\theta_1, \dots, \theta_{NJ} \in \{0, 1\}$.

661 As shown in Figure 7, the desired function ϕ can be computed by a Floor-ReLU
 662 network with width $2N$ and depth 4. Moreover, it holds that

663
$$\phi(s, n) = s_n, \quad \text{for } n = 1, 2, \dots, N.$$

664 So we finish the proof. □

665 The next lemma constructs a Floor-ReLU network that can extract any bit from a
 666 binary representation according to a specific index.

667 **Lemma 4.2.** *Given any $N, L \in \mathbb{N}^+$, there exists a function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ implemented by a*
 668 *Floor-ReLU network with width $2N + 2$ and depth $7L - 3$ such that, for any $\theta_m \in \{0, 1\}$,*
 669 *$m = 1, 2, \dots, N^L$, we have*

$$670 \quad \phi(\text{bin}0.\theta_1\theta_2\cdots\theta_{N^L}, m) = \theta_m, \quad \text{for } m = 1, 2, \dots, N^L.$$

671 *Proof.* The proof is based on repeated applications of Lemma 4.1. Specifically, we
 672 inductively construct a sequence of functions $\phi_1, \phi_2, \dots, \phi_L$ implemented by Floor-ReLU
 673 networks to satisfy the following two conditions for each $\ell \in \{1, 2, \dots, L\}$.

674 (i) $\phi_\ell : \mathbb{R}^2 \rightarrow \mathbb{R}$ can be implemented by a Floor-ReLU network with width $2N + 2$
 675 and depth $7\ell - 3$.

676 (ii) For any $\theta_m \in \{0, 1\}$, $m = 1, 2, \dots, N^\ell$, we have

$$677 \quad \phi_\ell(\text{bin}0.\theta_1\theta_2\cdots\theta_{N^\ell}, m) = \text{bin}0.\theta_m, \quad \text{for } m = 1, 2, \dots, N^\ell.$$

678 Firstly, consider the case $\ell = 1$. By Lemma 4.1 (set $J = 1$ therein), there exists a
 679 function ϕ_1 implemented by a Floor-ReLU network with width $2N \leq 2N + 2$ and depth
 680 $4 = 7 - 3$ such that, for any $\theta_m \in \{0, 1\}$, $m = 1, 2, \dots, N$, we have

$$681 \quad \phi_1(\text{bin}0.\theta_1\theta_2\cdots\theta_N, m) = \text{bin}0.\theta_m, \quad \text{for } m = 1, 2, \dots, N.$$

682 It follows that Condition (i) and (ii) hold for $\ell = 1$.

683 Next, assume Condition (i) and (ii) hold for $\ell = k$. We would like to construct ϕ_{k+1}
 684 to make Condition (i) and (ii) true for $\ell = k + 1$. By Lemma 4.1 (set $J = N^k$ therein),
 685 there exists a function ψ implemented by a Floor-ReLU network with width $2N$ and
 686 depth 4 such that, for any $\theta_m \in \{0, 1\}$, $m = 1, 2, \dots, N^{k+1}$, we have

$$687 \quad \psi(\text{bin}0.\theta_1\cdots\theta_{N^{k+1}}, n) = \text{bin}0.\theta_{(n-1)N^{k+1}}\cdots\theta_{(n-1)N^k+N^k}, \quad \text{for } n = 1, 2, \dots, N. \quad (10)$$

688 By the hypothesis of induction, we have

689 • $\phi_k : \mathbb{R}^2 \rightarrow \mathbb{R}$ can be implemented by a Floor-ReLU network with width $2N + 2$
 690 and depth $7k - 3$.

691 • For any $\theta_j \in \{0, 1\}$, $j = 1, 2, \dots, N^k$, we have

$$692 \quad \phi_k(\text{bin}0.\theta_1\theta_2\cdots\theta_{N^k}, j) = \text{bin}0.\theta_j, \quad \text{for } j = 1, 2, \dots, N^k. \quad (11)$$

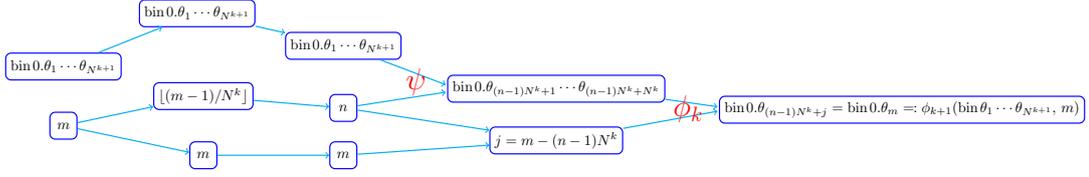


Figure 8: An illustration of the desired network architecture implementing ϕ_{k+1} based on (10), (11), and (12). We omit ReLU (σ) for neurons with non-negative inputs.

693 Given any $m \in \{1, 2, \dots, N^{k+1}\}$, there exist $n \in \{1, 2, \dots, N\}$ and $j \in \{1, 2, \dots, N^k\}$
 694 such that $m = (n-1)N^k + j$, and such n, j can be obtained by

$$695 \quad n = \lfloor (m-1)/N^k \rfloor + 1 \quad \text{and} \quad j = m - (n-1)N^k. \quad (12)$$

696 Then the desired architecture of the Floor-ReLU network implementing ϕ_{k+1} is shown
 697 in Figure 8.

698 Note that ψ can be computed by a Floor-ReLU network of width $2N$ and depth 4.

699 By Figure 8, we have

700 • $\phi_{k+1} : \mathbb{R}^2 \rightarrow \mathbb{R}$ can be implemented by a Floor-ReLU network with width $2N + 2$
 701 and depth $2 + 4 + 1 + (7k - 3) = 7(k + 1) - 3$, which implies Condition (i) for
 702 $\ell = k + 1$.

703 • For any $\theta_m \in \{0, 1\}$, $m = 1, 2, \dots, N^{k+1}$, we have

$$704 \quad \phi_{k+1}(\text{bin } 0.\theta_1\theta_2\cdots\theta_{N^{k+1}}, m) = \text{bin } 0.\theta_m, \quad \text{for } m = 1, 2, \dots, N^{k+1}.$$

705 That is, Condition (ii) holds for $\ell = k + 1$.

706 So we finish the process of induction.

707 By the principle of induction, there exists a function $\phi_L : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

708 • ϕ_L can be implemented by a Floor-ReLU network with width $2N + 2$ and depth
 709 $7L - 3$.

710 • For any $\theta_m \in \{0, 1\}$, $m = 1, 2, \dots, N^L$, we have

$$711 \quad \phi_L(\text{bin } 0.\theta_1\theta_2\cdots\theta_{N^L}, m) = \text{bin } 0.\theta_m, \quad \text{for } m = 1, 2, \dots, N^L.$$

712 Finally, define $\phi := 2\phi_L$. Then ϕ can also be implemented by a Floor-ReLU network
 713 with width $2N + 2$ and depth $7L - 3$. Moreover, for any $\theta_m \in \{0, 1\}$, $m = 1, 2, \dots, N^L$,
 714 we have

$$715 \quad \phi(\text{bin } 0.\theta_1\theta_2\cdots\theta_{N^L}, m) = 2 \cdot \phi_L(\text{bin } 0.\theta_1\theta_2\cdots\theta_{N^L}, m) = 2 \cdot \text{bin } 0.\theta_m = \theta_m,$$

716 for $m = 1, 2, \dots, N^L$. So we finish the proof. \square

717 With Lemma 4.2 in hand, we are ready to prove Proposition 3.2.

718 *Proof of Proposition 3.2.* By Lemma 4.2, there exists a function $\tilde{\phi} : \mathbb{R}^2 \rightarrow \mathbb{R}$ computed
 719 by a Floor-ReLU network with a fixed architecture with width $2N + 2$ and depth $7L - 3$
 720 such that, for any $z_m \in \{0, 1\}$, $m = 1, 2, \dots, N^L$, we have

$$721 \quad \tilde{\phi}(\text{bin}0.z_1z_2\cdots z_{N^L}, m) = z_m, \quad \text{for } m = 1, 2, \dots, N^L.$$

722 Based on $\theta_m \in \{0, 1\}$ for $m = 1, 2, \dots, N^L$ given in Proposition 3.2, we define the final
 723 function ϕ as

$$724 \quad \phi(x) := \tilde{\phi}(\sigma(x \cdot 0 + \text{bin}0.\theta_1\theta_2\cdots\theta_{N^L}), \sigma(x)), \quad \text{where } \sigma(x) = \max\{0, x\}.$$

725 Clearly, ϕ can be implemented by a Floor-ReLU network with width $2N + 2$ and depth
 726 $(7L - 3) + 1 = 7L - 2$. Moreover, we have, for any $m \in \{1, 2, \dots, N^L\}$,

$$727 \quad \phi(m) := \tilde{\phi}(\sigma(m \cdot 0 + \text{bin}0.\theta_1\theta_2\cdots\theta_{N^L}), \sigma(m)) = \tilde{\phi}(\text{bin}0.\theta_1\theta_2\cdots\theta_{N^L}, m) = \theta_m.$$

728 So we finish the proof. □

729 We finally point out that only the properties of Floor on $[0, \infty)$ are used in our proof.
 730 Thus, the Floor can be replaced by the truncation function that can be easily computed
 731 by truncating the decimal part.

732 5 Conclusion

733 This paper has introduced a theoretical framework to show that deep network approxi-
 734 mation can achieve root exponential convergence and avoid the curse of dimensionality
 735 for approximating functions as general as (Hölder) continuous functions. Given a Lip-
 736 schitz continuous function f on $[0, 1]^d$, it was shown by construction that Floor-ReLU
 737 networks with width $\max\{d, 5N + 13\}$ and depth $64dL + 3$ can achieve a uniform ap-
 738 proximation error bounded by $3\lambda\sqrt{d}N^{-\sqrt{L}}$, where λ is the Lipschitz constant of f .
 739 More generally for an arbitrary continuous function f on $[0, 1]^d$ with a modulus of con-
 740 tinuity $\omega_f(\cdot)$, the approximation error is bounded by $\omega_f(\sqrt{d}N^{-\sqrt{L}}) + 2\omega_f(\sqrt{d})N^{-\sqrt{L}}$.
 741 The results in this paper provide a theoretical lower bound of the power of deep network
 742 approximation. Whether or not this bound is achievable in actual computation relies on
 743 advanced algorithm design as a separate line of research.

744 **Acknowledgments.** Z. Shen is supported by Tan Chin Tuan Centennial Professor-
 745 ship. H. Yang was partially supported by the US National Science Foundation under
 746 award DMS-1945029.

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