
Neural Network Architecture Beyond Width and Depth

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Abstract

1 This paper proposes a new neural network architecture by introducing an additional
2 dimension called height beyond width and depth. Neural network architectures
3 with height, width, and depth as hyper-parameters are called three-dimensional
4 architectures. It is shown that neural networks with three-dimensional architectures
5 are significantly more expressive than the ones with two-dimensional architectures
6 (those with only width and depth as hyper-parameters), e.g., standard fully con-
7 nected networks. The new network architecture is constructed recursively via a
8 nested structure, and hence we call a network with the new architecture nested net-
9 work (NestNet). A NestNet of height s is built with each hidden neuron activated
10 by a NestNet of height $\leq s-1$. When $s=1$, a NestNet degenerates to a standard net-
11 work with a two-dimensional architecture. It is proved by construction that height- s
12 ReLU NestNets with $\mathcal{O}(n)$ parameters can approximate 1-Lipschitz continuous
13 functions on $[0, 1]^d$ with an error $\mathcal{O}(n^{-(s+1)/d})$, while the optimal approximation
14 error of standard ReLU networks with $\mathcal{O}(n)$ parameters is $\mathcal{O}(n^{-2/d})$. Further-
15 more, such a result is extended to generic continuous functions on $[0, 1]^d$ with
16 the approximation error characterized by the modulus of continuity. Finally, we
17 use numerical experimentation to show the advantages of the super-approximation
18 power of ReLU NestNets.

19 1 Introduction

20 In this paper, we design a new neural network architecture by introducing one more dimension, called
21 height, in addition to width and depth in the characterization of dimensions of neural networks. We
22 call neural network architectures with height, width, and depth as hyper-parameters three-dimensional
23 architectures. It is proved by construction that neural networks with three-dimensional architectures
24 improve the approximation power significantly, compared to standard networks with two-dimensional
25 architectures (those with only width and depth as hyper-parameters). The approximation power of
26 standard neural networks has been widely studied in recent years. The optimality of the approximation
27 of standard fully-connected rectified linear unit (ReLU) networks (e.g., see [35, 40, 49, 52]) implies
28 limited room for further improvements. This motivates us to design a new neural network architecture
29 by introducing an additional dimension of height beyond width and depth.

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30 We will focus on the ReLU ($\max\{0, x\}$) activation function and use it to demonstrate our ideas. Our
 31 new network architecture is constructed recursively via a nested structure, and hence we call a neural
 32 network with the new architecture nested network (**NestNet**). A NestNet of height s is built with each
 33 hidden neuron activated by a NestNet of height $\leq s - 1$. In the case of $s = 1$, a NestNet degenerates
 34 to a standard network with a two-dimensional architecture. Let us use a simple example to explain
 35 the height of a NestNet. We say a network is activated by $\varrho_1, \dots, \varrho_r$ if each hidden neuron of this
 36 network is activated by one of $\varrho_1, \dots, \varrho_r$. Here, $\varrho_1, \dots, \varrho_r$ are trainable functions mapping \mathbb{R} to \mathbb{R} .
 37 Then, a network of height $s \geq 2$ can be regarded as a $(\varrho_1, \dots, \varrho_r)$ -activated network, where $\varrho_1, \dots, \varrho_r$
 38 are (realized by) networks of height $\leq s - 1$. See an example of a height-2 network in Figure 1. The
 39 network therein can be regarded as a (ϱ_1, ϱ_2) -activated network, where ϱ_1 and ϱ_2 are (realized by)
 40 networks of height 1 (i.e., standard networks). The number of parameters in the network of Figure 1
 41 is the sum of the numbers of parameters in $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ and ϱ_1, ϱ_2 .

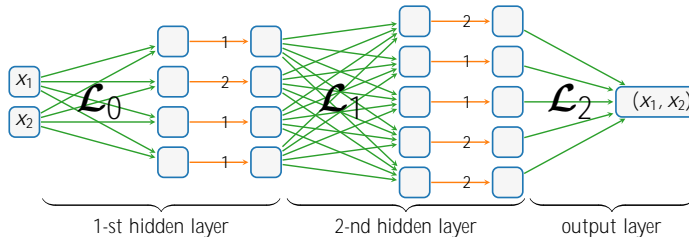


Figure 1: An example of a network of height 2, where ϱ_1 and ϱ_2 are (realized by) networks of height 1 (i.e., standard networks). Here, $\mathcal{L}_0, \mathcal{L}_1$ and \mathcal{L}_2 are affine linear maps.

42 We remark that a NestNet can be regarded as a sufficiently large standard network by expanding all
 43 of its sub-network activation functions. We propose the nested network architecture since it shares
 44 the parameters via repetitions of sub-network activation functions. In other words, a NestNet can
 45 provide a special parameter-sharing scheme. This is the key reason why the NestNet has much better
 46 approximation power than the standard network. If we regard the network in Figure 1 as a NestNet of
 47 height 2, then the number of parameters is the sum of the numbers of parameters in $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ and
 48 ϱ_1, ϱ_2 . However, if we expand the network in Figure 1 to a large standard network, then the number
 49 of parameters in ϱ_1 and ϱ_2 will be added many times for computing the total number of parameters.

50 Next, let us discuss our new network architecture from the perspective of hyper-parameters. We call
 51 the network architecture with only width as a hyper-parameter one-dimensional architecture. Its
 52 depth and height are both equal to one. Neural networks with this type of architecture are generally
 53 called shallow networks. See an example in Figure 2(a). We call the network architecture with
 54 only width and depth as hyper-parameters two-dimensional architecture. Its height is equal to one.
 55 Neural networks with this type of architecture are generally called deep networks. See an example
 56 in Figure 2(b). We call the network architecture with height, width, and depth as hyper-parameters
 57 three-dimensional architecture, which is proposed in this paper. Neural networks with this type of
 58 architecture are called NestNets. See an example in Figure 2(c). One may refer to Table 1 for the
 59 approximation power of networks with these three types of architectures discussed above.

Table 1: Comparison for the approximation error of 1-Lipschitz continuous functions on $[0, 1]^d$ approximated by ReLU NestNets and standard ReLU networks.

	dimension(s)	#parameters	approximation error	remark	reference
one-hidden-layer network	width varies (depth = height = 1)	$\mathcal{O}(n)$	n^{-1} for $d = 1$	linear combination	
deep network	width and depth vary (height = 1)	$\mathcal{O}(n)$	$n^{-2/d}$	composition	[35, 40, 49, 52]
NestNet of height s	width, depth, and height vary	$\mathcal{O}(n)$	$n^{-(s+1)/d}$	nested composition	this paper

60 Our main contributions are summarized as follows. We first propose a three-dimensional neural
 61 network architecture by introducing one more dimension called height beyond width and depth. We
 62 show that neural networks with three-dimensional architectures are significantly more expressive
 63 than standard networks. In particular, we prove that height- s ReLU NestNets with $\mathcal{O}(n)$ parameters
 64 can approximate 1-Lipschitz continuous functions on $[0, 1]^d$ with an error $\mathcal{O}(n^{-(s+1)/d})$, which is
 65 much better than the optimal error $\mathcal{O}(n^{-2/d})$ of standard ReLU networks with $\mathcal{O}(n)$ parameters. In

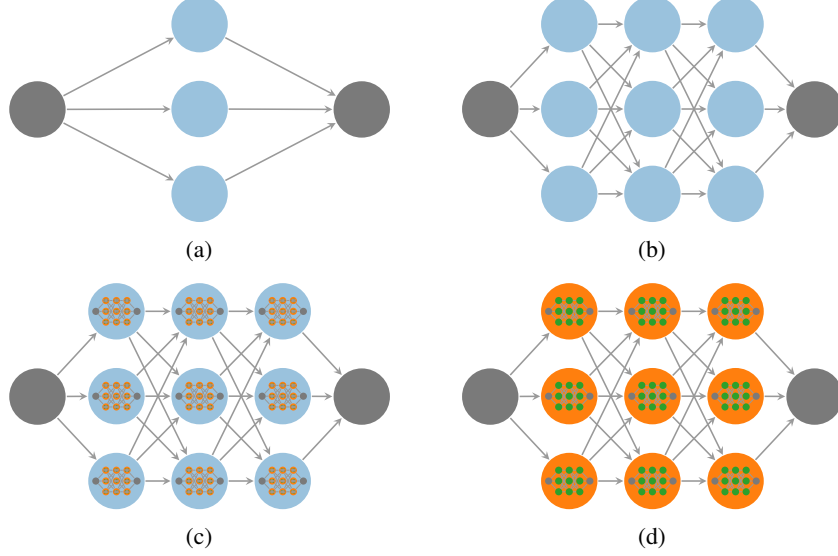


Figure 2: Illustrations of neural networks with one-, two-, and three-dimensional architectures. (a) One-dimensional case (width = 3, depth = height = 1). (b) Two-dimensional case (width = depth = 3, height = 1). (c) Three-dimensional case (width = depth = height = 3). (d) Zoom-in of an activation function of the network in (c). The network in (d) can also be regarded as a network of height 2.

66 the case of $s + 1 \geq d$, the approximation error is bounded by $\mathcal{O}(n^{-(s+1)/d}) \leq \mathcal{O}(n^{-1})$, which means
 67 we overcome the curse of dimensionality. Furthermore, we extend our result to generic continuous
 68 functions with the approximation error characterized by the modulus of continuity. See Theorem 2.1
 69 and Corollary 2.2 for more details. Finally, we conduct simple experiments to show the numerical
 70 advantages of the super-approximation power of ReLU NestNets.

71 The rest of this paper is organized as follows. In Section 2, we present the main results, provide the
 72 ideas of proving them, and discuss related work. The detailed proofs of the main results are placed
 73 in the appendix. Next, we conduct experiments to show the advantages of the super-approximation
 74 power of ReLU NestNets in Section 3. Finally, Section 4 concludes this paper with a short discussion.

75 2 Main results and related work

76 In this section, we first present our main results and discuss the proof ideas. The detailed proofs of the
 77 main results are placed in the appendix. Next, we discuss related work from multiple perspectives.

78 2.1 Main results

79 We use $\mathcal{NN}_s\{n\}$ for $n, s \in \mathbb{N}$ to denote the set of functions realized by height- s ReLU NestNets with
 80 as most n parameters. We will give the mathematical definition of $\mathcal{NN}_s\{n\}$. We first discuss some
 81 notations regarding affine linear maps. We use \mathcal{L} to denote the set of all affine linear maps, i.e.,

$$82 \quad \mathcal{L} := \left\{ \mathcal{L} : \mathcal{L}(\mathbf{x}) = \mathbf{W}\mathbf{x} + \mathbf{b}, \mathbf{W} \in \mathbb{R}^{d_2 \times d_1}, \mathbf{b} \in \mathbb{R}^{d_2}, d_1, d_2 \in \mathbb{N}^+ \right\}.$$

83 Let $\#\mathcal{L}$ denote the number of parameters in $\mathcal{L} \in \mathcal{L}$, i.e.,

$$84 \quad \#\mathcal{L} = (d_1 + 1)d_2 \quad \text{if } \mathcal{L}(\mathbf{x}) = \mathbf{W}\mathbf{x} + \mathbf{b} \quad \text{for } \mathbf{W} \in \mathbb{R}^{d_2 \times d_1} \text{ and } \mathbf{b} \in \mathbb{R}^{d_2}.$$

85 We use $\vec{g} = (\varrho_1, \dots, \varrho_k)$ to denote an activation function vector, where $\varrho_i : \mathbb{R} \rightarrow \mathbb{R}$ is an activation
 86 function for each $i \in \{1, \dots, k\}$. When $\vec{g} = (\varrho_1, \dots, \varrho_k)$ is applied to a vector input $\mathbf{x} = (x_1, \dots, x_k)$,

$$87 \quad \vec{g}(\mathbf{x}) = \left(\varrho_1(x_1), \dots, \varrho_k(x_k) \right) \quad \text{for any } \mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k.$$

88 Let $\text{set}(\vec{g})$ denote the function set containing all entries (functions) in \vec{g} . For example, if $\vec{g} =$
 89 $(\varrho_1, \varrho_2, \varrho_3, \varrho_2, \varrho_1)$, then $\text{set}(\vec{g}) = \{\varrho_1, \varrho_2, \varrho_3\}$.

90 To define $\mathcal{NN}_s\{n\}$ for $n, s \in \mathbb{N}$ recursively, we first consider the degenerate case. Define

$$91 \quad \mathcal{NN}_0\{n\} := \{\text{id}_{\mathbb{R}}, \text{ReLU}\} =: \mathcal{NN}_s\{0\} \quad \text{for } n, s \in \mathbb{N},$$

92 where $\text{id}_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}$ is the identity map. That is, we regard the identity map and ReLU as height-0
93 ReLU NestNets with n parameters or as height- s ReLU NestNets with 0 parameters.

94 Next, let us present the recursive step. For $n, s \in \mathbb{N}^+$, a (vector-valued) function $\phi \in \mathcal{NN}_s\{n\}$ has the
95 following form:

$$96 \quad \phi = \mathcal{L}_m \circ \vec{g}_m \circ \cdots \circ \mathcal{L}_1 \circ \vec{g}_1 \circ \mathcal{L}_0, \quad (1)$$

97 where $\mathcal{L}_0, \dots, \mathcal{L}_m \in \mathcal{L}$ are affine linear maps. Moreover, Equation (1) satisfies the following two
98 conditions:

99 • Condition on activation functions:

$$100 \quad \bigcup_{i=1}^m \text{set}(\vec{g}_i) = \{\varrho_1, \dots, \varrho_r\} \quad \text{and} \quad \varrho_j \in \bigcup_{i=0}^{s-1} \mathcal{NN}_i\{n_j\} \quad \text{for } j = 1, \dots, r. \quad (2)$$

101 Here, \vec{g}_i is an activation function vector for each $i \in \{1, \dots, m\}$. All entries in $\vec{g}_1, \dots, \vec{g}_m$
102 form an activation function set $\{\varrho_1, \dots, \varrho_r\}$. For each $j \in \{1, \dots, r\}$, ϱ_j can be realized by a
103 height- i NestNet with $\leq n_j$ parameters for some $i = i_j \leq s - 1$. This condition means each
104 hidden neuron is activated by a NestNet of lower height.

105 • Condition on the number of parameters:

$$106 \quad \sum_{i=0}^m \#\mathcal{L}_i + \sum_{j=1}^r n_j \leq n. \quad (3)$$

107 This condition means the total number of parameters is no more than n . The total number of
108 parameters is calculated by adding two parts. The first one is the number of parameters in
109 affine linear maps $\mathcal{L}_0, \dots, \mathcal{L}_m$. The other part is the number of parameters in the activation
110 set $\{\varrho_1, \dots, \varrho_r\}$ formed by the entries in activation function vectors $\vec{g}_1, \dots, \vec{g}_m$.

111 Then, with two conditions in Equations (2) and (3), we can define $\mathcal{NN}_s\{n\}$ for $n, s \in \mathbb{N}^+$ as follows:

$$112 \quad \mathcal{NN}_s\{n\} := \left\{ \begin{array}{l} \phi = \mathcal{L}_m \circ \vec{g}_m \circ \cdots \circ \mathcal{L}_1 \circ \vec{g}_1 \circ \mathcal{L}_0, \quad \mathcal{L}_0, \dots, \mathcal{L}_m \in \mathcal{L}, \quad \bigcup_{i=1}^m \text{set}(\vec{g}_i) = \{\varrho_1, \dots, \varrho_r\}, \\ \varrho_j \in \bigcup_{i=0}^{s-1} \mathcal{NN}_i\{n_j\} \quad \text{for } j = 1, \dots, r, \quad \sum_{i=0}^m \#\mathcal{L}_i + \sum_{j=1}^r n_j \leq n \end{array} \right\}.$$

113 We remark that, in the definition above, m can be equal to 0. In this case, the function ϕ degenerates
114 to an affine linear map.

115 In the NestNet example in Figure 1, the function ϕ therein is in $\bigcup_{n \in \mathbb{N}} \mathcal{NN}_2\{n\}$ and the activation
116 function vectors \vec{g}_1 and \vec{g}_2 can be represented as

$$117 \quad \vec{g}_1 = (\varrho_1, \varrho_2, \varrho_1, \varrho_1) \quad \text{and} \quad \vec{g}_2 = (\varrho_2, \varrho_1, \varrho_1, \varrho_2, \varrho_2).$$

118 Moreover, the activation function set containing all entries in \vec{g}_1 and \vec{g}_2 is a subset of $\bigcup_{n \in \mathbb{N}} \mathcal{NN}_1\{n\}$,
119 i.e., $\{\varrho_1, \varrho_2\} \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{NN}_1\{n\}$.

120 Let $C([0, 1]^d)$ denote the set of continuous functions on $[0, 1]^d$. By convention, the modulus of
121 continuity of a continuous function $f \in C([0, 1]^d)$ is defined as

$$122 \quad \omega_f(r) := \sup \{ \|f(\mathbf{x}) - f(\mathbf{y})\| : \|\mathbf{x} - \mathbf{y}\|_2 \leq r, \mathbf{x}, \mathbf{y} \in [0, 1]^d \} \quad \text{for any } r \geq 0.$$

123 Under these settings, we can find a function in $\mathcal{NN}_s\{\mathcal{O}(n)\}$ to approximate $f \in C([0, 1]^d)$ with an
124 approximation error $\mathcal{O}(\omega_f(n^{-(s+1)/d}))$, as shown in the main theorem below.

125 **Theorem 2.1.** *Given a continuous function $f \in C([0, 1]^d)$, for any $n, s \in \mathbb{N}^+$ and $p \in [1, \infty]$, there
126 exists $\phi \in \mathcal{NN}_s\{C_{s,d}(n+1)\}$ such that*

$$127 \quad \|\phi - f\|_{L^p([0,1]^d)} \leq 7\sqrt{d}\omega_f(n^{-(s+1)/d}),$$

128 where $C_{s,d} = 10^3 d^2 (s+7)^2$ if $p \in [1, \infty)$ and $C_{s,d} = 10^{d+3} d^2 (s+7)^2$ if $p = \infty$.

129 We remark that the constant $C_{s,d}$ in Theorem 2.1 is valid for all $n \in \mathbb{N}^+$. As we shall see later, $C_{s,d}$
130 can be greatly reduced if one only cares about large $n \in \mathbb{N}^+$. Generally, it is challenging to simplify
131 the approximation error in Theorem 2.1 to make it explicitly depend on n due to the complexity of
132 $\omega_f(\cdot)$. However, the approximation error can be simplified to an explicit one depending on n in the
133 case of special target function spaces like Hölder continuous function space. To be exact, if f is a
134 Hölder continuous function on $[0, 1]^d$ of order $\alpha \in (0, 1]$ with a Hölder constant $\lambda > 0$, then

$$135 \quad |f(\mathbf{x}) - f(\mathbf{y})| \leq \lambda \|\mathbf{x} - \mathbf{y}\|_2^\alpha \quad \text{for any } \mathbf{x}, \mathbf{y} \in [0, 1]^d,$$

136 implying $\omega_f(r) \leq \lambda r^\alpha$ for any $r \geq 0$. This means we can get an exponentially small approximation
137 error $7\lambda\sqrt{d}n^{-(s+1)\alpha/d}$ as shown in Corollary 2.2 below.

138 **Corollary 2.2.** *Suppose f is a Hölder continuous function on $[0, 1]^d$ of order $\alpha \in (0, 1]$ with a*
139 *Hölder constant $\lambda > 0$. For any $n, s \in \mathbb{N}^+$ and $p \in [1, \infty]$, there exists $\phi \in \mathcal{NN}_s\{C_{s,d}(n+1)\}$ such*
140 *that*

$$141 \quad \|\phi - f\|_{L^p([0,1]^d)} \leq 7\lambda\sqrt{d}n^{-(s+1)\alpha/d},$$

142 where $C_{s,d} = 10^3 d^2 (s+7)^2$ if $p \in [1, \infty)$ and $C_{s,d} = 10^{d+3} d^2 (s+7)^2$ if $p = \infty$.

143 In Corollary 2.2, if $\alpha = 1$, i.e., f is a Lipschitz continuous function with a Lipschitz constant
144 $\lambda > 0$, then the approximation error can be further simplified to $7\lambda\sqrt{d}n^{-(s+1)/d}$. See Table 1 for the
145 comparison of the approximation error of 1-Lipschitz continuous functions on $[0, 1]^d$ approximated
146 by ReLU NestNets and standard ReLU networks.

147 2.2 Sketch of proving Theorem 2.1

148 We will discuss how to prove Theorem 2.1. Given a target function $f \in C([0, 1]^d)$, the key point is
149 to construct an almost piecewise constant function realized by a ReLU NestNet to approximate f
150 well except for a small region. Then we can get the desired result by dealing with the approximation
151 in this small region. We divide the sketch of proving Theorem 2.1 into three main steps.

152 1. First, we divide $[0, 1]^d$ into a union of cubes $\{Q\}_{\mathbf{i} \in \{0,1,\dots,K-1\}^d}$ and a small region Ω with
153 $K = \mathcal{O}(n^{(s+1)/d})$. Each Q is associated with a representative $\mathbf{x} \in Q$ for each vector index \mathbf{i} .
154 See Figure 3 for an illustration for $K = 4$ and $d = 2$.

155 2. Next, we design a vector-valued function $\Phi_1(\mathbf{x})$ to map the whole cube Q to its index \mathbf{i} for
156 each \mathbf{i} . Here, Φ_1 can be defined/constructed via

$$157 \quad \Phi_1(\mathbf{x}) = [\phi_1(x_1), \phi_1(x_2), \dots, \phi_1(x_d)]^T,$$

158 where each one-dimensional function ϕ_1 is a step function outside a small region. We can
159 efficiently construct ReLU NestNets with the desired size to approximate such an almost step
160 function ϕ_1 with sufficiently many ‘‘steps’’ by using the composition architecture of ReLU
161 NestNets. See the appendix for the detailed construction.

162 3. Finally, we need to construct a function ϕ_2 realized by a ReLU NestNet to map \mathbf{i} approximately
163 to $f(\mathbf{x}(\mathbf{i}))$ for each $\mathbf{i} \in \{0, 1, \dots, K-1\}^d$. Then we have

$$164 \quad \phi_2 \circ \Phi_1(\mathbf{x}) = \phi_2(\mathbf{i}) \approx f(\mathbf{x}(\mathbf{i})) \approx f(\mathbf{x}) \quad \text{for any } \mathbf{x} \in Q \text{ and each } \mathbf{i},$$

165 implying

$$166 \quad \phi := \phi_2 \circ \Phi_1 \approx f \quad \text{on } [0, 1]^d \setminus \Omega.$$

167 Then, we can get a good approximation on $[0, 1]^d$ by using Lemma 3.4 of our previous paper [24]
168 to deal with the approximation inside Ω . We remark that, in the construction of $\phi_2 : \mathbb{R}^d \rightarrow \mathbb{R}$, we
169 only need to care about the values of ϕ_2 at a set of K^d points $\{0, 1, \dots, K-1\}^d$. As we shall see
170 later, this is the key point to ease the design of a ReLU NestNet with the desired size to realize ϕ_2 .

171 See Figure 3 for an illustration of the above steps. Observe that in Figure 3, we have

$$172 \quad \phi(\mathbf{x}) = \phi_2 \circ \Phi_1(\mathbf{x}) = \phi_2(\mathbf{i}) \stackrel{E_1}{\approx} f(\mathbf{x}(\mathbf{i})) \stackrel{E_2}{\approx} f(\mathbf{x})$$

173 for any $\mathbf{x} \in Q$ and each $\mathbf{i} \in \{0, 1, \dots, K-1\}^d$. That means $\phi - f$ is bounded by $E_1 + E_2$ on $[0, 1]^d \setminus \Omega$.
174 For any $\mathbf{x} \in Q$ and each \mathbf{i} , we have

$$175 \quad \|\mathbf{x}(\mathbf{i}) - \mathbf{x}\|_2 \leq \sqrt{d}/K \implies |f(\mathbf{x}(\mathbf{i})) - f(\mathbf{x})| \leq \omega_f(\sqrt{d}/K) \implies E_2 \leq \omega_f(\sqrt{d}/K).$$

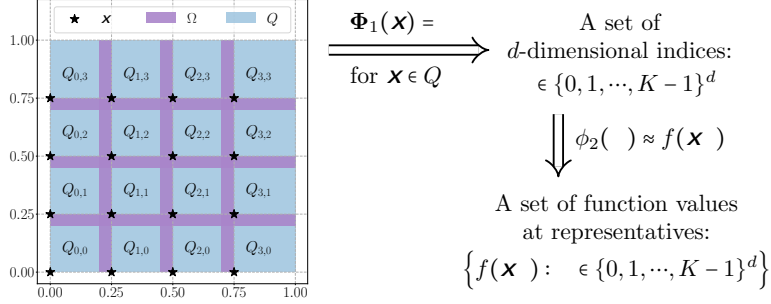


Figure 3: An illustration of the ideas of constructing $\phi = \phi_2 \circ \Phi_1$ to approximate f for $K = 4$ and $d = 2$. Note that $\phi \approx f$ outside Ω since $\phi(\mathbf{x}) = \phi_2 \circ \Phi_1(\mathbf{x}) = \phi_2(\cdot) \approx f(\mathbf{x}) \approx f(\mathbf{x})$ for any $\mathbf{x} \in Q$ and each $\cdot \in \{0, 1, \dots, K - 1\}^d$.

176 The upper bound of E_1 is determined by the construction of $\phi_2 : \mathbb{R}^d \rightarrow \mathbb{R}$. As stated previously, we
 177 only need to care about the values of ϕ_2 at a set of K^d points $\{0, 1, \dots, K - 1\}^d \subseteq \mathbb{R}^d$, which gives us
 178 much freedom to control E_1 . As we shall see later, E_1 can be bounded by $\mathcal{O}(\omega_f(\sqrt{d}/K))$. Therefore,
 179 $\phi - f$ is controlled by $\mathcal{O}(\omega_f(\sqrt{d}/K))$ outside Ω , from which we deduce the desired approximation
 180 error on $[0, 1]^d \setminus \Omega$ since $K = \mathcal{O}(n^{-(s+1)/d})$. Finally, by using Lemma 3.4 of our previous paper [24]
 181 to deal with the approximation inside Ω , we can get the desired approximation error on $[0, 1]^d$.

182 2.3 Related work

183 We first compare our results with existing ones from an approximation perspective. Next, we discuss
 184 the parameter-sharing schemes of neural networks. Finally, we connect our NestNet architecture to
 185 existing trainable activation functions.

186 Discussion from an approximation perspective

187 The study of the approximation power of deep neural networks has become an active topic in recent
 188 years. This topic has been extensively studied from many perspectives, e.g., in terms of combinatorics
 189 [27], topology [7], information theory [29], fat-shattering dimension [1, 21], Vapnik-Chervonenkis
 190 (VC) dimension [6, 14, 32], classical approximation theory [3, 4, 8, 9, 10, 11, 12, 13, 18, 22, 24, 25,
 191 28, 34, 35, 38, 39, 42, 48, 49, 52, 53], etc. To the best of our knowledge, the study of neural network
 192 approximation has two main stages: shallow (one-hidden-layer) networks and deep networks.

193 In the early works of neural network approximation, the approximation power of shallow networks is
 194 investigated. In particular, the universal approximation theorem [11, 17, 18], without approximation
 195 error estimate, showed that a sufficiently large neural network can approximate a target function
 196 in a certain function space arbitrarily well. For one-hidden-layer neural networks of width n and
 197 sufficiently smooth functions, an asymptotic approximation error $\mathcal{O}(n^{-1/2})$ in the L^2 -norm is proved
 198 in [4, 5], leveraging an idea that is similar to Monte Carlo sampling for high-dimensional integrals.

199 Recently, a large number of works focus on the study of deep neural networks. It is shown in
 200 [35, 49, 52] that the optimal approximation error is $\mathcal{O}(n^{-2/d})$ by using ReLU networks with n
 201 parameters to approximate 1-Lipschitz continuous functions on $[0, 1]^d$. This optimal approximation
 202 error follows a natural question: How can we get a better approximation error? Generally, there
 203 are two ideas to get better errors. The first one is to consider smaller function spaces, e.g., smooth
 204 functions [24, 50] and band-limited functions [26]. The other one is to introduce new networks,
 205 e.g., Floor-ReLU networks [36], Floor-Exponential-Step (FLES) networks [37], and (Sin, ReLU,
 206 2^x)-activated networks [20].

207 This paper proposes a three-dimensional neural network architecture by introducing one more
 208 dimension called height beyond width and depth. As shown in Theorem 2.1 and Corollary 2.2, neural
 209 networks with three-dimensional architectures are significantly more expressive than the ones with
 210 two-dimensional architectures. We will conduct experiments to explore the numerical properties of
 211 NestNets in Section 3.

212 Discussion from a parameter-sharing perspective

213 As discussed previously, our NestNet architecture can be regarded as a sufficiently large standard
214 network architecture with a specific parameter-sharing scheme. Parameter-sharing schemes are
215 used in neural networks to control the overall number of parameters for reducing memory and
216 communication costs. There are two common parameter-sharing schemes for a neural network. The
217 first scheme is to share parameters in the same layer. A typical network example with this scheme is
218 the convolutional neural network (CNN). In CNN architectures, filters in a CNN layer are shared for
219 all channels, which means the parameters in the filters are shared. The second scheme is to share
220 parameters across different layers of networks, e.g., recurrent neural networks.

221 In the NestNet architecture, we share parameters via repetitions of sub-network activation functions.
222 Both of parameter-sharing schemes discussed just above are used in the NestNet architecture. The
223 nested architecture of NestNets gives us much freedom to determine how many parameters to share.
224 Beyond parameter-sharing schemes for a neural network, there are also parameter-sharing schemes
225 among different neural networks or models, especially for multi-task learning. One may refer to
226 [30, 33, 44, 45, 46, 51] for more discussion on parameter sharing in neural networks.

227 Connection to trainable activation functions

228 The key idea of trainable activation functions is to add a small number of trainable parameters to
229 existing activation functions. Let us present several existing trainable activation functions as follows.
230 A ReLU-like function is introduced in [15] by modifying the negative part of ReLU using a trainable
231 parameter α , i.e., the parametric ReLU (PReLU) is defined as $\text{PReLU}(x) := \begin{cases} x & \text{if } x \geq 0 \\ \alpha x & \text{if } x < 0. \end{cases}$ A variant
232 of ELU unit is introduced in [43] by adding two trainable parameters $\beta, \gamma > 0$, i.e., the parametric
233 ELU (PELU) is given by $\text{PELU}(x) := \begin{cases} x & \text{if } x \geq 0 \\ (\exp(\gamma x) - 1)x & \text{if } x < 0. \end{cases}$ Authors in [31] propose a type of
234 flexible ReLU (FReLU), which is defined via $\text{FReLU}(x) := \text{ReLU}(x + \alpha) + \beta$, where α and β are two
235 trainable parameters. One may refer to [2] for a survey of modern trainable activation functions. To
236 the best of our knowledge, most existing trainable activation functions can be regarded as parametric
237 variants of the original activation functions. That is, they are attained via parameterizing the original
238 activation functions with a small number of (typically 1 or 2) trainable parameters.

239 By contrast, activation functions in our NestNets are much more flexible. They can be (realized
240 by) either complicated or simple sub-NestNets. That is, we can freely determine the number of
241 parameters in the activation functions of NestNets. In other words, in NestNets, we can randomly
242 distribute the parameters in the affine linear maps and activation functions. In short, compared to the
243 networks with existing trainable activation functions, our NestNets are more flexible and have much
244 more freedom in the choice of activation functions.

245 3 Experimentation

246 In this section, we will conduct experiments as a proof of concept to explore the numerical properties
247 of ReLU NestNets. It is challenging to tune the hyper-parameters of large NestNets due to their
248 nested architectures. Thus, our experimentation focuses on relatively small NestNets of height
249 2 and we introduce a simple sub-network activation function ϱ , which is realized by a trainable
250 one-hidden-layer ReLU network of width 3. To be exact, ϱ is given by

$$251 \quad \varrho(x) = \mathbf{w}_1^T \cdot (x\mathbf{w}_0 + \mathbf{b}_0) + b_1 \quad \text{for any } x \in \mathbb{R}, \quad (4)$$

252 where $\mathbf{w}_0, \mathbf{w}_1, \mathbf{b}_0 \in \mathbb{R}^3$ and $b_1 \in \mathbb{R}$ are trainable parameters. There are 10 parameters in ϱ . The initial
253 settings for ϱ in our experiments are $\mathbf{w}_0 = (1, 1, 1)$, $\mathbf{w}_1 = (1, 1, -1)$, $\mathbf{b}_0 = (-0.2, -0.1, 0.0)$, and
254 $b_1 = 0$. We believe that NestNets can achieve good results in some real-world applications if proper
255 optimization algorithms are developed for NestNets. In this paper, we only consider two classification
256 problems: a synthetic classification problem based on the Archimedean spiral in Section 3.1 and an
257 image classification problem corresponding to a standard benchmark dataset Fashion-MNIST [47]
258 in Section 3.2. We remark that a classification function can be continuously extended to \mathbb{R}^d if each
259 class of samples are located in a bounded closed subset of \mathbb{R}^d and these subsets are pairwise disjoint.
260 That means we can apply our theory to classification problems.

261 3.1 Archimedean spiral

262 We will design a binary classification experiment by constructing two disjoint sets based on the
 263 Archimedean spiral, which can be described by the equation $r = a + b\theta$ in polar coordinates (r, θ) for
 264 given $a, b \in \mathbb{R}$. Let us first define two curves (Archimedean spirals) as follows:

$$265 \quad \tilde{\mathcal{C}}_i := \left\{ (x, y) : x = r_i \cos \theta, y = r_i \sin \theta, r_i = a_i + b_i \theta, \theta \in [0, s\pi] \right\},$$

266 for $i = 0, 1$, where $a_0 = 0, a_1 = 1, b_0 = b_1 = 1/\pi$, and $s = 30$. To simplify the discussion below, we
 267 normalize $\tilde{\mathcal{C}}_i$ as $\mathcal{C}_i \subseteq [0, 1]^2$, where \mathcal{C}_i is defined by

$$268 \quad \mathcal{C}_i := \left\{ (x, y) : x = \frac{\tilde{x}}{2(s+2)} + \frac{1}{2}, y = \frac{\tilde{y}}{2(s+2)} + \frac{1}{2}, (\tilde{x}, \tilde{y}) \in \tilde{\mathcal{C}}_i \right\},$$

269 for $i = 0, 1$. Then, we can define the two desired sets as follows:

$$270 \quad \mathcal{S}_i := \left\{ (u, v) : \sqrt{(u-x)^2 + (v-y)^2} \leq \varepsilon, (x, y) \in \mathcal{C}_i \right\},$$

271 for $i = 0, 1$, where $\varepsilon = 0.005$ in our experiments. See an illustration for \mathcal{S}_0 and \mathcal{S}_1 in Figure 4.

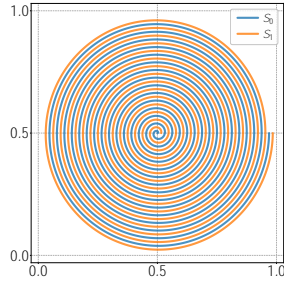


Figure 4: An illustration for \mathcal{S}_0 and \mathcal{S}_1 .

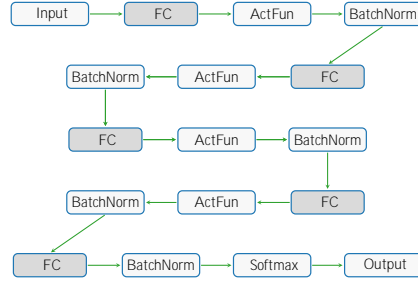


Figure 5: A network architecture illustration.

272 To explore the numerical performance of NestNets, we design NestNets and standard networks to
 273 classify samples in $\mathcal{S}_0 \cup \mathcal{S}_1$. We adopt four-hidden-layer fully connected network architecture of
 274 width 20, 35, or 50. To make the optimization more stable, we add the layers of batch normalization
 275 [19]. See Figure 5 for an illustration of the full network architecture. In Figure 5, FC and ActFun
 276 are short of fully connected layer and activation function, respectively. ActFun is ReLU for standard
 277 networks, while for NestNets, ActFun is the learnable sub-network activation function ϱ given in
 278 Equation (4).

279 Before presenting the experiment results, let us present the hyper-parameters for training the networks
 280 mentioned above. For each $i \in \{0, 1\}$, we randomly choose 3×10^5 training samples and 3×10^4 test
 281 samples in \mathcal{S}_i with label i . Then, we use these 6×10^5 training samples to train the networks and use
 282 these 6×10^4 test samples to compute the test accuracy. We use the cross-entropy loss function to
 283 evaluate the loss between the networks and the target classification function. The number of epochs
 284 and the batch size are set to 500 and 512, respectively. We adopt RAdam [23] as the optimization
 285 method. In epochs $5(i-1) + 1$ to $5i$ for $i = 1, 2, \dots, 100$, the learning rate is $0.2 \times 0.002 \times 0.9^{i-1}$
 286 for the parameters in ϱ and $0.002 \times 0.9^{i-1}$ for all other parameters. We remark that all training (test)
 287 samples are standardized before training, i.e., we rescale the samples to have a mean of 0 and a
 288 standard deviation of 1.

289 Finally, let us present the experiment results to compare the numerical performances of NestNets
 290 and standard networks. We adopt the average of test accuracies in the last 100 epochs as the target
 291 test accuracy. As we can see from Table 2 and Figure 6, by adding 10 more parameters (stored in ϱ),
 292 NestNets achieve much better test accuracies than standard networks though slightly more training
 293 time is required. In an “unfair” comparison, the test accuracy attained by the NestNet with 1.4×10^3
 294 parameters is still better than that of the standard network with 7.9×10^3 parameters. This numerically
 295 verifies that the NestNet has much better approximation power than the standard network.

296 3.2 Fashion-MNIST

297 We will design convolutional neural network (CNN) architectures activated by ReLU or the sub-
 298 network activation function ϱ given in Equation (4) to classify image samples in Fashion-MNIST [47].

Table 2: Test accuracy comparison.

	width	depth	#parameters	activation function	training time	test accuracy
standard network	20	4	1362	ReLU	≈ 2532 s	0.738290
NestNet	20	4	1362 + 10	sub-network (ϱ)	≈ 4016 s	0.873631
standard network	35	4	3957	ReLU	≈ 2595 s	0.816048
NestNet	35	4	3957 + 10	sub-network (ϱ)	≈ 4104 s	0.995962
standard network	50	4	7902	ReLU	≈ 2642 s	0.866118
NestNet	50	4	7902 + 10	sub-network (ϱ)	≈ 4218 s	0.999984

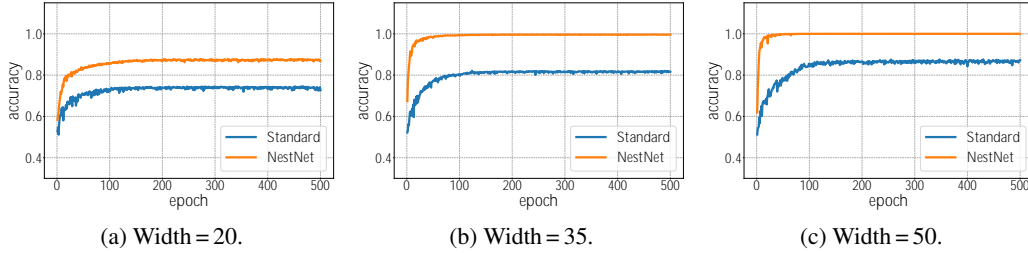


Figure 6: Test accuracy over epochs.

299 This dataset consists of a training set of 6×10^4 samples and a test set of 10^4 samples. Each sample
 300 is a 28×28 grayscale image, associated with a label from 10 classes. To compare the numerical
 301 performances of NestNets and standard networks, we design a standard CNN architecture and a
 302 NestNet architecture that is constructed by replacing a few activation functions of a standard CNN
 303 network by the sub-network activation function ϱ . For simplicity, we denote the standard CNN and
 304 the NestNet as CNN1 and CNN2. To make the optimization more stable, we add the layers of dropout
 305 [16, 41] and batch normalization [19]. See illustrations of CNN1 and CNN2 in Figure 7. We present
 306 more details of them in Table 3.

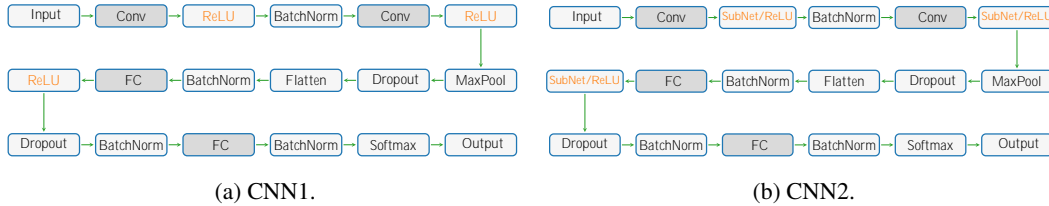


Figure 7: Illustrations of CNN1 and CNN2. Conv and FC represent convolutional and fully connected layers, respectively. CNN2 is indeed a NestNet of height 2.

Table 3: Details of CNN1 and CNN2.

layers	activation function		output size of each layer	dropout	batch normalization
	CNN1	CNN2			
input $\in \mathbb{R}^{28 \times 28}$			28×28		
Conv-1: $1 \times (3 \times 3)$, 12	ReLU	SubNet (ϱ), ReLU,	$1 \times (26 \times 26)$ $11 \times (26 \times 26)$		yes
Conv-2: $12 \times (3 \times 3)$, 12	ReLU	SubNet (ϱ), ReLU,	$1 \times (24 \times 24)$ $11 \times (24 \times 24)$	1728 (MaxPool & Flatten)	0.25 yes
FC-1: 1728, 48	ReLU	SubNet (ϱ), ReLU,	1 47	48	0.5 yes
FC-2: 48, 10			10 (Softmax)		yes
output $\in \mathbb{R}^{10}$					

307 Before presenting the numerical results, let us present the hyper-parameters for training two CNN
 308 architectures above. We use the cross-entropy loss function to evaluate the loss between the CNNs
 309 and the target classification function. The number of epochs and the batch size are set to 500 and 128,
 310 respectively. We adopt RADam [23] as the optimization method and the weight decay of the optimizer
 311 is 0.0001. In epochs $5(i - 1) + 1$ to $5i$ for $i = 1, 2, \dots, 100$, the learning rate is $0.2 \times 0.002 \times 0.9^{i-1}$

312 for the parameters in ϱ and $0.002 \times 0.9^{i-1}$ for all other parameters. All training (test) samples in the
 313 Fashion-MNIST dataset are standardized in our experiment, i.e., we rescale all training (test) samples
 314 to have a mean of 0 and a standard deviation of 1. In the settings above, we repeat the experiment
 315 18 times and discard 3 top-performing and 3 bottom-performing trials by using the average of test
 316 accuracy in the last 100 epochs as the performance criterion. For each epoch, we adopt the average of
 317 test accuracies in the rest 12 trials as the target test accuracy.

318 Next, let us present the experiment results to compare the numerical performances of CNN1 and
 319 CNN2. The test accuracy comparison of CNN1 and CNN2 is summarized in Table 4.

Table 4: Test accuracy comparison.

	training time	largest accuracy	average of largest 100 accuracies	average accuracy in last 100 epochs
CNN1	≈ 5802 s	0.925290	0.924796	0.924447
CNN2	≈ 7217 s	0.926620	0.926287	0.926032

320 For each of CNN1 and CNN2, we present the training time, the largest test accuracy, the average
 321 of the largest 100 test accuracies, and the average of test accuracies in the last 100 epochs. For an
 322 intuitive comparison, we also provide illustrations of the test accuracy over epochs for CNN1 and
 323 CNN2 in Figure 8. As we can see from Table 4 and Figure 8, CNN2 performs better than CNN1
 324 though slightly more training time and 10 more parameters are required. This numerically shows that
 325 the NestNet is significantly more expressive than the standard network.

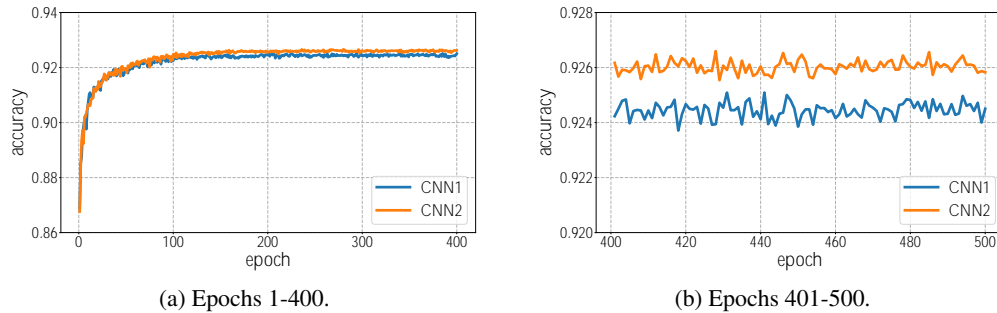


Figure 8: Test accuracy over epochs.

326 4 Conclusion

327 This paper proposes a three-dimensional neural network architecture by introducing one more
 328 dimension called height beyond width and depth. We show by construction that neural networks with
 329 three-dimensional architectures are significantly more expressive than the ones with two-dimensional
 330 architectures. We use simple numerical examples to show the advantages of the super-approximation
 331 power of ReLU NestNets, which is regarded as a proof of possibility. It would be of great interest to
 332 further explore the numerical performance of NestNets to bridge our theoretical results to applications.
 333 We believe that NestNets can be further developed and applied to real-world applications.

334 We remark that our analysis is limited to the ReLU activation function and the (Hölder) continuous
 335 function space. It would be interesting to generalize our results to other activation functions (e.g.,
 336 tanh and sigmoid functions) and other function spaces (e.g., Lebesgue and Sobolev spaces).

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497 **A Proof of main theorem**

498 In this section, we will prove the main theorem, Theorem 2.1, based on an auxiliary theorem,
 499 Theorem A.1, which will be proved in Section B. Notations throughout this paper are summarized in
 500 Section A.1.

501 **A.1 Notations**

502 Let us summarize all basic notations used in this paper as follows.

- 503 • Let \mathbb{R} , \mathbb{Q} , and \mathbb{Z} denote the set of real numbers, rational numbers, and integers, respectively.
- 504 • Let \mathbb{N} and \mathbb{N}^+ denote the set of natural numbers and positive natural numbers, respectively.
 505 That is, $\mathbb{N}^+ = \{1, 2, 3, \dots\}$ and $\mathbb{N} = \mathbb{N}^+ \cup \{0\}$.
- 506 • For any $x \in \mathbb{R}$, let $\lfloor x \rfloor := \max\{n : n \leq x, n \in \mathbb{Z}\}$ and $\lceil x \rceil := \min\{n : n \geq x, n \in \mathbb{Z}\}$.
- 507 • Let 1_S be the indicator (characteristic) function of a set S , i.e., 1_S is equal to 1 on S and 0
 508 outside S .
- 509 • The set difference of two sets A and B is denoted by $A \setminus B := \{x : x \in A, x \notin B\}$.
- 510 • Matrices are denoted by bold uppercase letters. For instance, $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a real matrix
 511 of size $m \times n$ and \mathbf{A}^T denotes the transpose of \mathbf{A} . Vectors are denoted as bold lowercase
 512 letters. For example, $\mathbf{v} = [v_1, \dots, v_d]^T = \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} \in \mathbb{R}^d$ is a column vector.
- 513 • For any $p \in [1, \infty)$, the p -norm (or ℓ^p -norm) of a vector $\mathbf{x} = [x_1, x_2, \dots, x_d]^T \in \mathbb{R}^d$ is defined
 514 by

$$\|\mathbf{x}\|_p = \|\mathbf{x}\|_{\ell^p} := (|x_1|^p + |x_2|^p + \dots + |x_d|^p)^{1/p}.$$

516 In the case of $p = \infty$,

$$\|\mathbf{x}\|_\infty = \|\mathbf{x}\|_{\ell^\infty} := \max\{|x_i| : i = 1, 2, \dots, d\}.$$

- 518 • By convention, $\sum_{j=n_1}^{n_2} a_j = 0$ if $n_1 > n_2$, no matter what a_j is for each j .
- 519 • Given any $K \in \mathbb{N}^+$ and $\delta \in (0, \frac{1}{K})$, define a trifling region $\Omega([0, 1]^d, K, \delta)$ of $[0, 1]^d$ as

$$\Omega([0, 1]^d, K, \delta) := \bigcup_{j=1}^d \left\{ \mathbf{x} = [x_1, x_2, \dots, x_d]^T \in [0, 1]^d : x_j \in \bigcup_{k=1}^{K-1} \left(\frac{k}{K} - \delta, \frac{k}{K} \right) \right\}. \quad (5)$$

521 In particular, $\Omega([0, 1]^d, K, \delta) = \emptyset$ if $K = 1$. See Figure 9 for two examples of trifling
 522 regions.

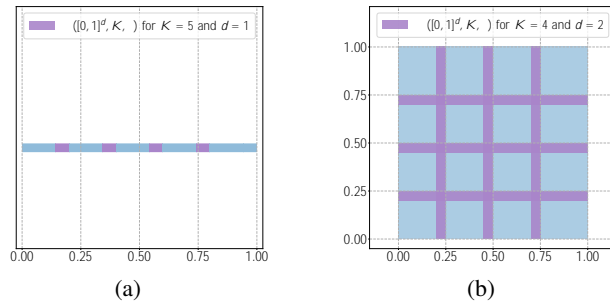


Figure 9: Two examples of trifling regions. (a) $K = 5, d = 1$. (b) $K = 4, d = 2$.

- 523 • For a continuous piecewise linear function $f(x)$, the x values where the slope changes are
 524 typically called **breakpoints**.
- 525 • Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ denote the rectified linear unit (ReLU), i.e. $\sigma(x) = \max\{0, x\}$ for any $x \in \mathbb{R}$.
 526 With a slight abuse of notation, we define $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ as $\sigma(\mathbf{x}) = \begin{bmatrix} \max\{0, x_1\} \\ \vdots \\ \max\{0, x_d\} \end{bmatrix}$ for any
 527 $\mathbf{x} = [x_1, \dots, x_d]^T \in \mathbb{R}^d$.

528 • Let $\mathcal{NN}_s\{n\}$ for $n, s \in \mathbb{N}^+$ denote the set of functions realized by height- s ReLU NestNets
 529 with as most n parameters.

530 • A function ϕ realized by a ReLU network can be briefly described as follows:

531
$$\mathbf{x} = \tilde{\mathbf{h}}_0 \xrightarrow[\mathcal{L}_0]{W_0, b_0} \mathbf{h}_1 \xrightarrow{\sigma} \tilde{\mathbf{h}}_1 \cdots \xrightarrow[\mathcal{L}_{L-1}]{W_{L-1}, b_{L-1}} \mathbf{h}_L \xrightarrow{\sigma} \tilde{\mathbf{h}}_L \xrightarrow[\mathcal{L}_L]{W_L, b_L} \mathbf{h}_{L+1} = \phi(\mathbf{x}),$$

532 where $W_i \in \mathbb{R}^{N_{i+1} \times N_i}$ and $b_i \in \mathbb{R}^{N_{i+1}}$ are the weight matrix and the bias vector in the i -th
 533 affine linear transformation \mathcal{L}_i , respectively, i.e.,

534
$$\mathbf{h}_{i+1} = W_i \cdot \tilde{\mathbf{h}}_i + b_i =: \mathcal{L}_i(\tilde{\mathbf{h}}_i) \quad \text{for } i = 0, 1, \dots, L,$$

535 and

536
$$\tilde{\mathbf{h}}_i = \sigma(\mathbf{h}_i) \quad \text{for } i = 1, 2, \dots, L.$$

537 In particular, ϕ can be represented in a form of function compositions as follows

538
$$\phi = \mathcal{L}_L \circ \sigma \circ \cdots \circ \mathcal{L}_1 \circ \sigma \circ \mathcal{L}_0,$$

539 which has been illustrated in Figure 10.

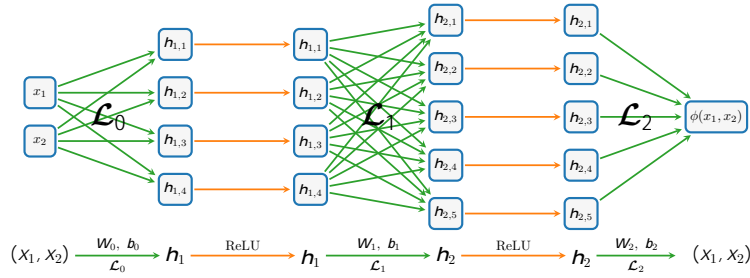


Figure 10: An example of a ReLU network of width 5 and depth 2.

- 540 • The expression “a network of width N and depth L ” means
- 541 – The number of neurons in each **hidden** layer of this network (architecture) is no more
 - 542 than N .
 - 543 – The number of **hidden** layers of this network (architecture) is no more than L .

544 A.2 Detailed proof of Theorem 2.1

545 The key point of proving Theorem 2.1 is to construct a piecewise constant function to approximate
 546 the target continuous function. However, ReLU NestNets are unable to approximate piecewise
 547 constant functions well the continuity of ReLU NestNets. Thus, we introduce the trifling region
 548 $\Omega([0, 1]^d, K, \delta)$, defined in Equation (5), and use ReLU NestNets to implement piecewise constant
 549 functions outside the trifling region. To simplify the proof of Theorem 2.1, we introduce an auxiliary
 550 theorem, Theorem A.1 below. It can be regarded as a weaker variant of Theorem 2.1, ignoring the
 551 approximation in the trifling region.

552 **Theorem A.1.** *Given a continuous function $f \in C([0, 1]^d)$, for any $n, s \in \mathbb{N}^+$, there exists $\phi \in$
 553 $\mathcal{NN}_s\{355d^2(s+7)^2(2n+1)\}$ such that $\|\phi\|_{L^\infty(\mathbb{R}^d)} \leq |f(\mathbf{0})| + \omega_f(\sqrt{d})$ and*

554
$$|\phi(\mathbf{x}) - f(\mathbf{x})| \leq 6\sqrt{d}\omega_f(n^{-(s+1)/d}) \quad \text{for any } \mathbf{x} \in [0, 1]^d \setminus \Omega([0, 1]^d, K, \delta),$$

555 where $K = \lfloor n^{(s+1)/d} \rfloor$ and δ is an arbitrary number in $(0, \frac{1}{3K}]$.

556 The proof of Theorem A.1 can be found in Section B. By assuming Theorem A.1 is true, we can
 557 easily prove Theorem 2.1 for the case $p \in [1, \infty)$. To prove Theorem 2.1 for the case $p = \infty$, we need
 558 to control the approximation error in the trifling region. To this intent, we introduce a theorem to
 559 handle the approximation inside the trifling region.

560 **Theorem A.2** (Lemma 3.11 of [52] or Lemma 3.4 of [24]). Given any $\varepsilon > 0$, $K \in \mathbb{N}^+$, and $\delta \in (0, \frac{1}{3K}]$,
 561 assume $f \in C([0, 1]^d)$ and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is a general function with

$$562 \quad |g(\mathbf{x}) - f(\mathbf{x})| \leq \varepsilon \quad \text{for any } \mathbf{x} \in [0, 1]^d \setminus \Omega([0, 1]^d, K, \delta).$$

563 Then

$$564 \quad |\phi(\mathbf{x}) - f(\mathbf{x})| \leq \varepsilon + d \cdot \omega_f(\delta) \quad \text{for any } \mathbf{x} \in [0, 1]^d,$$

565 where $\phi := \phi_d$ is defined by induction through $\phi_0 := g$ and

$$566 \quad \phi_{i+1}(\mathbf{x}) := \text{mid}(\phi_i(\mathbf{x} - \delta \mathbf{e}_{i+1}), \phi_i(\mathbf{x}), \phi_i(\mathbf{x} + \delta \mathbf{e}_{i+1})) \quad \text{for } i = 0, 1, \dots, d-1,$$

567 where $\{\mathbf{e}_i\}_{i=1}^d$ is the standard basis in \mathbb{R}^d and $\text{mid}(\cdot, \cdot, \cdot)$ is the function returning the middle value of
 568 three inputs.

569 Now, let us prove Theorem 2.1 by assuming Theorem A.1 is true, the proof of which can be found in
 570 Section B.

571 *Proof of Theorem 2.1.* We may assume f is not a constant function since it is a trivial case. Then
 572 $\omega_f(r) > 0$ for any $r > 0$. Let us first consider the case $p \in [1, \infty)$. Set $K = \lfloor n^{(s+1)/d} \rfloor$ and choose a
 573 sufficiently small $\delta \in (0, \frac{1}{3K}]$ such that

$$574 \quad \begin{aligned} Kd\delta(2|f(\mathbf{0})| + 2\omega_f(\sqrt{d}))^p &= \lfloor n^{(s+1)/d} \rfloor d\delta(2|f(\mathbf{0})| + 2\omega_f(\sqrt{d}))^p \\ &\leq \left(\omega_f(n^{-(s+1)/d})\right)^p. \end{aligned}$$

575 By Theorem A.1, there exists

$$576 \quad \begin{aligned} \phi &\in \mathcal{NN}_s\{355d^2(s+7)^2(2n+1)\} \subseteq \mathcal{NN}_s\{355d^2(s+7)^2 \cdot 2(n+1)\} \\ &\subseteq \mathcal{NN}_s\{10^3d^2(s+7)^2(n+1)\} \end{aligned}$$

577 such that $\|\phi\|_{L^\infty(\mathbb{R}^d)} \leq |f(\mathbf{0})| + \omega_f(\sqrt{d})$ and

$$578 \quad |\phi(\mathbf{x}) - f(\mathbf{x})| \leq 6\sqrt{d}\omega_f(n^{-(s+1)/d}) \quad \text{for any } \mathbf{x} \in [0, 1]^d \setminus \Omega([0, 1]^d, K, \delta).$$

579 Since $\|f\|_{L^\infty([0, 1]^d)} \leq |f(\mathbf{0})| + \omega_f(\sqrt{d})$ and the Lebesgue measure of $\Omega([0, 1]^d, K, \delta)$ is bounded
 580 by $Kd\delta$, we have

$$581 \quad \begin{aligned} \|\phi - f\|_{L^p([0, 1]^d)}^p &= \int_{\Omega([0, 1]^d, K, \delta)} |\phi(\mathbf{x}) - f(\mathbf{x})|^p d\mathbf{x} + \int_{[0, 1]^d \setminus \Omega([0, 1]^d, K, \delta)} |\phi(\mathbf{x}) - f(\mathbf{x})|^p d\mathbf{x} \\ &\leq Kd\delta(2|f(\mathbf{0})| + 2\omega_f(\sqrt{d}))^p + \left(6\sqrt{d}\omega_f(n^{-(s+1)/d})\right)^p \\ &\leq \left(\omega_f(n^{-(s+1)/d})\right)^p + \left(6\sqrt{d}\omega_f(n^{-(s+1)/d})\right)^p \leq \left(7\sqrt{d}\omega_f(n^{-(s+1)/d})\right)^p. \end{aligned}$$

582 Hence, we have $\|\phi - f\|_{L^p([0, 1]^d)} \leq 7\sqrt{d}\omega_f(n^{-(s+1)/d})$.

583 Next, let us discuss the case $p = \infty$. Set $K = \lfloor n^{(s+1)/d} \rfloor$ and choose a sufficiently small $\delta \in (0, \frac{1}{3K}]$
 584 such that

$$585 \quad d \cdot \omega_f(\delta) \leq \omega_f(n^{-(s+1)/d}).$$

586 By Theorem A.1,

$$587 \quad \phi_0 \in \mathcal{NN}_s\{355d^2(s+7)^2(2n+1)\}$$

588 such that

$$589 \quad |\phi_0(\mathbf{x}) - f(\mathbf{x})| \leq 6\sqrt{d}\omega_f(n^{-(s+1)/d}) \quad \text{for any } \mathbf{x} \in [0, 1]^d \setminus \Omega([0, 1]^d, K, \delta).$$

590 By Theorem A.2 with $g = \phi_0$ and $\varepsilon = 6\sqrt{d}\omega_f(n^{-(s+1)/d})$ therein, we have

$$591 \quad |\phi(\mathbf{x}) - f(\mathbf{x})| \leq \varepsilon + d \cdot \omega_f(\delta) \leq 7\sqrt{d}\omega_f(n^{-(s+1)/d}) \quad \text{for any } \mathbf{x} \in [0, 1]^d,$$

592 where $\phi := \phi_d$ is defined by induction through

$$593 \quad \phi_{i+1}(\mathbf{x}) := \text{mid}(\phi_i(\mathbf{x} - \delta \mathbf{e}_{i+1}), \phi_i(\mathbf{x}), \phi_i(\mathbf{x} + \delta \mathbf{e}_{i+1})) \quad \text{for } i = 0, 1, \dots, d-1,$$

594 where $\{\mathbf{e}_i\}_{i=1}^d$ is the standard basis in \mathbb{R}^d and $\text{mid}(\cdot, \cdot, \cdot)$ is the function returning the middle value of
 595 three inputs.

596 It remains to estimate the number of parameters in the NestNet realizing $\phi = \phi_d$. By Lemma 3.1 of
 597 [37], $\text{mid}(\cdot, \cdot, \cdot)$ can be realized by a ReLU network of width 14 and depth 2, and hence with at most
 598 $14 \times (14 + 1) \times (2 + 1) = 630$ parameters.

599 By defining a vector-valued function $\Phi_0 : \mathbb{R}^d \rightarrow \mathbb{R}^3$ as

$$600 \quad \Phi_0(\mathbf{x}) := [\phi_0(\mathbf{x} - \delta \mathbf{e}_1), \phi_0(\mathbf{x}), \phi_0(\mathbf{x} + \delta \mathbf{e}_1)]^T \quad \text{for any } \mathbf{x} \in \mathbb{R}^d,$$

601 we have $\Phi_0 \in \mathcal{NN}_s\{3^2(355d^2(s+7)^2(2n+1))\}$, implying

$$602 \quad \begin{aligned} \phi_1 = \text{mid}(\cdot, \cdot, \cdot) \circ \Phi_0 &\in \mathcal{NN}_s\left\{630 + 3^2(355d^2(s+7)^2(2n+1))\right\} \\ &\subseteq \mathcal{NN}_s\left\{10(355d^2(s+7)^2(2n+1))\right\}. \end{aligned}$$

603 Similarly, we have

$$604 \quad \begin{aligned} \phi = \phi_d \in \mathcal{NN}_s\left\{10^d(355d^2(s+7)^2(2n+1))\right\} &\subseteq \mathcal{NN}_s\left\{10^d(355d^2(s+7)^2 \cdot 2(n+1))\right\} \\ &\subseteq \mathcal{NN}_s\left\{10^{d+3}d^2(s+7)^2(n+1)\right\}. \end{aligned}$$

605 Thus, we finish the proof of Theorem 2.1.

606 □

607 B Proof of auxiliary theorem

608 We will prove the auxiliary theorem, Theorem A.1, in this section. We first present the key ideas
 609 in Section B.1. Next, the detailed proof is presented in Section B.2, based on two propositions in
 610 Section B.1, the proofs of which can be found in Sections C and D.

611 B.1 Key ideas of proving Theorem A.1

612 Our goal is to construct an almost piecewise constant function realized by a ReLU NestNet to
 613 approximate the target function $f \in C([0, 1]^d)$ well. The construction can be divided into three main
 614 steps.

615 1. First, we divide $[0, 1]^d$ into a union of “important” cubes $\{Q\}_{Q \in \{0,1,\dots,K-1\}^d}$ and the trifling
 616 region $\Omega([0, 1]^d, K, \delta)$, where $K = \mathcal{O}(n^{(s+1)/d})$. Each Q is associated with a representative
 617 $\mathbf{x} \in Q$ for each vector index \cdot . See Figure 13 for illustrations.

618 2. Next, we design a vector-valued function $\Phi_1(\mathbf{x})$ to map the whole cube Q to its index \cdot for
 619 each \cdot . Here, Φ_1 can be defined/constructed via

$$620 \quad \Phi_1(\mathbf{x}) = [\phi_1(x_1), \phi_1(x_2), \dots, \phi_1(x_d)]^T,$$

621 where each one-dimensional function ϕ_1 is a step function outside the trifling region and hence
 622 can be realized by a ReLU NestNet.

623 3. The aim of the final step is essentially to solve a point fitting problem. We will construct a function
 624 ϕ_2 realized by a ReLU NestNet to map \cdot approximately to $f(\mathbf{x} \cdot)$ for each \cdot . Then we have

$$625 \quad \phi_2 \circ \Phi_1(\mathbf{x}) = \phi_2(\cdot) \approx f(\mathbf{x} \cdot) \approx f(\mathbf{x}) \quad \text{for any } \mathbf{x} \in Q \text{ and each } \cdot,$$

626 implying

$$627 \quad \phi := \phi_2 \circ \Phi_1 \approx f \quad \text{on } [0, 1]^d \setminus \Omega([0, 1]^d, K, \delta).$$

628 We remark that, in the construction of ϕ_2 , we only need to care about the values of ϕ_2 sampled
 629 inside the set $\{0, 1, \dots, K-1\}^d$, which is a key point to ease the design of a ReLU NestNet to
 630 realize ϕ_2 as we shall see later.

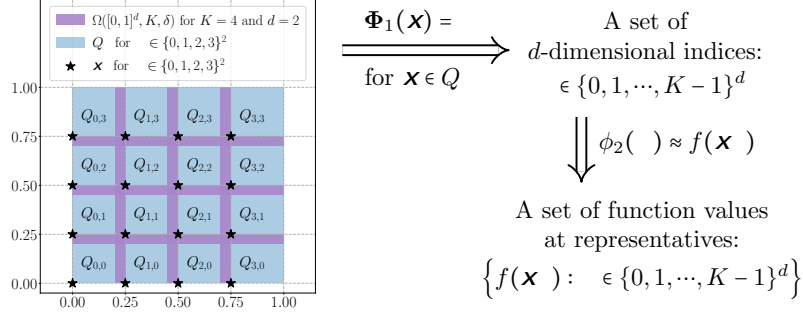


Figure 11: An illustration of the ideas of constructing the desired function $\phi = \phi_2 \circ \Phi_1$. Note that $\phi \approx f$ outside the trifling region since $\phi(\mathbf{x}) = \phi_2 \circ \Phi_1(\mathbf{x}) = \phi_2(\cdot) \approx f(\mathbf{x}) \approx f(\mathbf{x})$ for any $\mathbf{x} \in Q$ and each $\cdot \in \{0, 1, \dots, K-1\}^d$.

631 Observe that in Figure 11, we have

$$632 \quad \phi(\mathbf{x}) = \phi_2 \circ \Phi_1(\mathbf{x}) = \phi_2(\cdot) \stackrel{E_1}{\approx} f(\mathbf{x}) \stackrel{E_2}{\approx} f(\mathbf{x})$$

633 for any $\mathbf{x} \in Q$ and each $\cdot \in \{0, 1, \dots, K-1\}^d$. That means $\phi - f$ is controlled by $E_1 + E_2$ on
634 $[0, 1]^d \setminus \Omega([0, 1]^d, K, \delta)$. Since $\|\mathbf{x} - \cdot\|_2 \leq \sqrt{d}/K$ for any $\mathbf{x} \in Q$ and each \cdot , E_2 is bounded by
635 $\omega_f(\sqrt{d}/K)$. As we shall see later, E_1 can be bounded by $\mathcal{O}(\omega_f(\sqrt{d}/K))$ by applying Proposi-
636 tion B.2. Therefore, $\phi - f$ is controlled by $\mathcal{O}(\omega_f(\sqrt{d}/K))$ outside the trifling region, from which
637 we deduce the desired approximation error since $K = \mathcal{O}(n^{-(s+1)/d})$.

638 Finally, we introduce two propositions to simplify the constructions of Φ_1 and ϕ_2 mentioned above.
639 We first show how to construct a ReLU network to implement a one-dimensional step function ϕ_1 in
640 Proposition B.1 below. Then Φ_1 can be defined via

$$641 \quad \Phi_1(\mathbf{x}) := [\phi_1(x_1), \phi_1(x_2), \dots, \phi_1(x_d)]^T \quad \text{for any } \mathbf{x} = [x_1, x_2, \dots, x_d]^T \in \mathbb{R}^d.$$

642 **Proposition B.1.** *Given any $n, r \in \mathbb{N}^+$, $\delta \in (0, 1)$, and $J \in \mathbb{N}^+$ with $J \leq 2^{n^r}$, there exists $\phi \in$
643 $\mathcal{NN}_r\{36(r+7)n\}$ such that*

$$644 \quad \phi(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{j=0}^{J-1} [j, j+1-\delta]$$

645 and

$$646 \quad \phi(x) = J \quad \text{for any } x \in [J, J+1].$$

647 The construction of ϕ_2 is mainly based on Proposition B.2 below, whose proof relies on the bit
648 extraction technique proposed in [6]. As we shall see later, some pre-processing is necessary for
649 meeting the requirements of applying Proposition B.2 to construct ϕ_2 .

650 **Proposition B.2.** *Given any $\varepsilon > 0$ and $n, s \in \mathbb{N}^+$, assume $y_j \geq 0$ for $j = 0, 1, \dots, J-1$ are samples
651 with $J \leq n^{s+1}$ and*

$$652 \quad |y_j - y_{j-1}| \leq \varepsilon \quad \text{for } j = 1, 2, \dots, J-1.$$

653 *Then there exists $\phi \in \mathcal{NN}_s\{350(s+7)^2(n+1)\}$ such that*

$$654 \quad (i) \quad |\phi(j) - y_j| \leq \varepsilon \quad \text{for } j = 0, 1, \dots, J-1.$$

$$655 \quad (ii) \quad 0 \leq \phi(x) \leq \max\{y_j : j = 0, 1, \dots, J-1\} \quad \text{for any } x \in \mathbb{R}.$$

656 The proofs of these two propositions can be found in Sections C and D. We will give the detailed
657 proof of Theorem A.1 in Section B.2.

658 **B.2 Detailed proof of Theorem A.1**

659 We essentially construct an almost piecewise constant function realized by a ReLU NestNet with
 660 at most $\mathcal{O}(n)$ parameters to approximate f . We may assume f is not a constant function since
 661 it is a trivial case. Then $\omega_f(r) > 0$ for any $r > 0$. It is clear that $|f(\mathbf{x}) - f(\mathbf{0})| \leq \omega_f(\sqrt{d})$ for
 662 any $\mathbf{x} \in [0, 1]^d$. By defining $\tilde{f} := f - f(\mathbf{0}) + \omega_f(\sqrt{d})$, we have $\omega_{\tilde{f}}(r) = \omega_f(r)$ for any $r \geq 0$ and
 663 $0 \leq \tilde{f}(\mathbf{x}) \leq 2\omega_f(\sqrt{d})$ for any $\mathbf{x} \in [0, 1]^d$.

664 Set $K = \lfloor n^{(s+1)/d} \rfloor$ and let δ be an arbitrary number in $(0, \frac{1}{3K}]$. The proof can be divided into four
 665 main steps as follows:

- 666 1. Divide $[0, 1]^d$ into a union of sub-cubes $\{Q\}_{\epsilon \in \{0,1,\dots,K-1\}^d}$ and the trifling region
 667 $\Omega([0, 1]^d, K, \delta)$, and denote \mathbf{x} as the vertex of Q with minimum $\|\cdot\|_1$ norm.
- 668 2. Construct a sub-network based on Proposition B.1 to implement a vector function Φ_1
 669 projecting the whole cube Q to the d -dimensional index for each ϵ , i.e., $\Phi_1(\mathbf{x}) = \epsilon$ for
 670 all $\mathbf{x} \in Q$.
- 671 3. Construct a sub-network to implement a function ϕ_2 mapping the index ϵ approximately to
 672 $\tilde{f}(\mathbf{x})$. This core step can be further divided into three sub-steps:
 - 673 3.1. Construct a sub-network to implement ψ_1 bijectively mapping the index set
 674 $\{0, 1, \dots, K-1\}^d$ to an auxiliary set $\mathcal{A}_1 \subseteq \{\frac{j}{2K^d} : j = 0, 1, \dots, 2K^d\}$ defined later.
 675 See Figure 14 for an illustration.
 - 676 3.2. Determine a continuous piecewise linear function g with a set of breakpoints $\mathcal{A}_1 \cup$
 677 $\mathcal{A}_2 \cup \{1\}$, where $\mathcal{A}_2 \subseteq \{\frac{j}{2K^d} : j = 0, 1, \dots, 2K^d\}$ is a set defined later. Moreover, g
 678 should satisfy two conditions: 1) the values of g at breakpoints in \mathcal{A}_1 is given based on
 679 $\{\tilde{f}(\mathbf{x})\}$, i.e., $g \circ \psi_1(\epsilon) = \tilde{f}(\mathbf{x})$; 2) the values of g at breakpoints in $\mathcal{A}_2 \cup \{1\}$ is
 680 defined to reduce the variation of g , which is necessary for applying Proposition B.2.
 - 681 3.3. Apply Proposition B.2 to construct a sub-network to implement a function ψ_2 approxi-
 682 mating g well on $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \{1\}$. Then the desired function ϕ_2 is given by $\phi_2 = \psi_2 \circ \psi_1$
 683 satisfying $\phi_2(\epsilon) = \psi_2 \circ \psi_1(\epsilon) \approx g \circ \psi_1(\epsilon) = \tilde{f}(\mathbf{x})$.
- 684 4. Construct the final network to implement the desired function ϕ via $\phi = \phi_2 \circ \Phi_1 + f(\mathbf{0}) -$
 685 $\omega_f(\sqrt{d})$. Then we have $\phi_2 \circ \Phi_1(\mathbf{x}) = \phi_2(\epsilon) \approx \tilde{f}(\mathbf{x}) \approx \tilde{f}(\mathbf{x})$ for any $\mathbf{x} \in Q$ and
 686 $\epsilon \in \{0, 1, \dots, K-1\}^d$, implying $\phi(\mathbf{x}) = \phi_2 \circ \Phi_1(\mathbf{x}) + f(\mathbf{0}) - \omega_f(\sqrt{d}) \approx \tilde{f}(\mathbf{x}) + f(\mathbf{0}) -$
 687 $\omega_f(\sqrt{d}) = f(\mathbf{x})$.

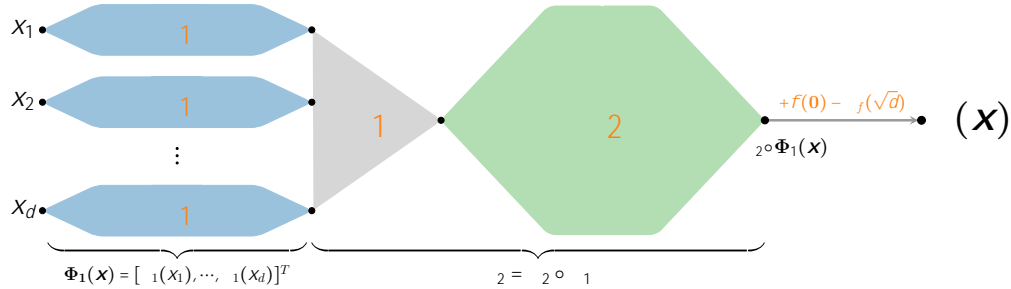


Figure 12: An illustration of the NestNet architecture realizing $\phi = \phi_2 \circ \Phi_1 + f(\mathbf{0}) - \omega_f(\sqrt{d})$. Here, ϕ_1 is implemented via Proposition B.1; $\psi_1 : \mathbb{R}^d \rightarrow \mathbb{R}$ is an affine linear function; ψ_2 is implemented via Proposition B.2.

688 See Figure 12 for an illustration of the NestNet architecture realizing $\phi = \phi_2 \circ \Phi_1 + f(\mathbf{0}) - \omega_f(\sqrt{d})$.
 689 The details of the steps mentioned above can be found below.

690 **Step 1:** Divide $[0, 1]^d$ into $\{Q\}_{\epsilon \in \{0,1,\dots,K-1\}^d}$ and $\Omega([0, 1]^d, K, \delta)$.

691 Define $\mathbf{x} := \lfloor K$ and

$$692 \quad Q := \left\{ \mathbf{x} = [x_1, x_2, \dots, x_d]^T \in [0, 1]^d : x_i \in \left[\frac{\beta_i}{K}, \frac{\beta_i+1}{K} - \delta \cdot \mathbf{1}_{\{\beta_i \leq K-2\}} \right], \quad i = 1, 2, \dots, d \right\}$$

693 for each d -dimensional index $\mathbf{c} = [\beta_1, \beta_2, \dots, \beta_d]^T \in \{0, 1, \dots, K-1\}^d$. Recall that $\Omega([0, 1]^d, K, \delta)$ is
 694 the trifling region defined in Equation (5). Apparently, $\mathbf{x} = \lfloor K$ is the vertex of Q with minimum
 695 $\|\cdot\|_1$ norm and

$$696 \quad [0, 1]^d = \left(\bigcup_{\mathbf{c} \in \{0, 1, \dots, K-1\}^d} Q_{\mathbf{c}} \right) \cup \Omega([0, 1]^d, K, \delta).$$

697 See Figure 13 for illustrations.

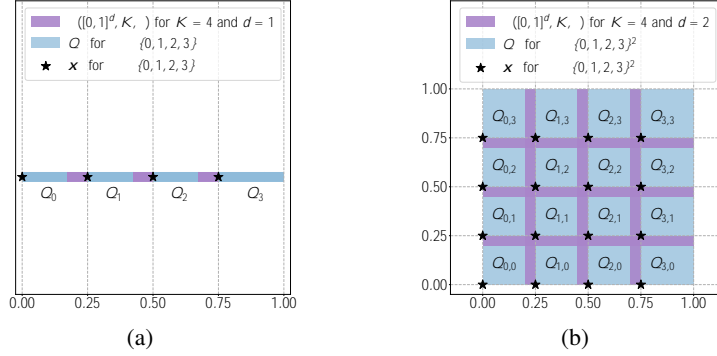


Figure 13: Illustrations of $\Omega([0, 1]^d, K, \delta)$, $Q_{\mathbf{c}}$, and $\mathbf{x}_{\mathbf{c}}$ for $\mathbf{c} \in \{0, 1, \dots, K-1\}^d$. (a) $K = 4$ and $d = 1$. (b) $K = 4$ and $d = 2$.

698 **Step 2:** Construct Φ_1 mapping $\mathbf{x} \in Q_{\mathbf{c}}$ to \mathbf{y} .

699 Note that

$$700 \quad K-1 = \lfloor n^{(s+1)/d} \rfloor - 1 \leq n^{s+1} \leq (n^s)^2 \leq 4^{(n^s)} = 2^{2(n^s)} \leq 2^{(2n)^s} = 2^{\tilde{n}^s},$$

701 where $\tilde{n} = 2n$. By Proposition B.1 with $r = s$ and $J = K-1 \leq 2^{\tilde{n}^s} = 2^{\tilde{n}^r}$ therein, there exists

$$702 \quad \tilde{\phi}_1 \in \mathcal{NN}_s\{36(s+7)\tilde{n}\} = \mathcal{NN}_s\{36(s+7)(2n)\} = \mathcal{NN}_s\{72(s+7)n\}$$

703 such that

$$704 \quad \tilde{\phi}_1(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{k=0}^{K-2} [k, k+1 - \tilde{\delta}] \text{ with } \tilde{\delta} = K\delta$$

705 and

$$706 \quad \tilde{\phi}_1(x) = K-1 \quad \text{for any } x \in [K-1, K].$$

707 Define $\phi_1(x) := \tilde{\phi}_1(Kx)$ for any $x \in \mathbb{R}$. Then, we have $\phi_1 \in \mathcal{NN}_s\{72(s+7)n\}$ and

$$708 \quad \phi_1(x) = k \quad \text{if } x \in \left[\frac{k}{K}, \frac{k+1}{K} - \delta \cdot \mathbf{1}_{\{k \leq K-2\}} \right] \quad \text{for } k = 0, 1, \dots, K-1.$$

709 It follows that $\phi_1(x_i) = \beta_i$ if $\mathbf{x} = [x_1, x_2, \dots, x_d]^T \in Q_{\mathbf{c}}$ for each $\mathbf{c} = [\beta_1, \beta_2, \dots, \beta_d]^T$.

710 By defining

$$711 \quad \Phi_1(\mathbf{x}) := [\phi_1(x_1), \phi_1(x_2), \dots, \phi_1(x_d)]^T \quad \text{for any } \mathbf{x} = [x_1, x_2, \dots, x_d]^T \in \mathbb{R}^d,$$

712 we have

$$713 \quad \Phi_1(\mathbf{x}) = \mathbf{c} \quad \text{if } \mathbf{x} \in Q_{\mathbf{c}} \quad \text{for each } \mathbf{c} \in \{0, 1, \dots, K-1\}^d. \quad (6)$$

714 **Step 3:** Construct ϕ_2 mapping \mathbf{x} approximately to $\tilde{f}(\mathbf{x})$.

715 The construction of the sub-network implementing ϕ_2 is essentially based on Proposition B.2. To
 716 meet the requirements of applying Proposition B.2, we first define two auxiliary sets \mathcal{A}_1 and \mathcal{A}_2 as

$$717 \quad \mathcal{A}_1 := \left\{ \frac{i}{K^{d-1}} + \frac{k}{2K^d} : i = 0, 1, \dots, K^{d-1} - 1 \quad \text{and} \quad k = 0, 1, \dots, K-1 \right\}$$

718 and

719
$$\mathcal{A}_2 := \left\{ \frac{i}{K^{d-1}} + \frac{K+k}{2K^d} : i = 0, 1, \dots, K^{d-1}-1 \quad \text{and} \quad k = 0, 1, \dots, K-1 \right\}.$$

720 Clearly,

721
$$\mathcal{A}_1 \cup \mathcal{A}_2 \cup \{1\} = \left\{ \frac{j}{2K^d} : j = 0, 1, \dots, 2K^d \right\} \quad \text{and} \quad \mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset.$$

722 See Figure 13 for an illustration of \mathcal{A}_1 and \mathcal{A}_2 . Next, we further divide this step into three sub-steps.

723 **Step 3.1:** Construct ψ_1 bijectively mapping $\{0, 1, \dots, K-1\}^d$ to \mathcal{A}_1 .

724 Inspired by the binary representation, we define

725
$$\psi_1(\mathbf{x}) := \frac{x_d}{2K^d} + \sum_{i=1}^{d-1} \frac{x_i}{K^i} \quad \text{for any } \mathbf{x} = [x_1, x_2, \dots, x_d]^T \in \mathbb{R}^d. \quad (7)$$

726 Then ψ_1 is a linear function bijectively mapping the index set $\{0, 1, \dots, K-1\}^d$ to

727
$$\begin{aligned} \left\{ \psi_1(\mathbf{x}) : \mathbf{x} \in \{0, 1, \dots, K-1\}^d \right\} &= \left\{ \frac{\beta_d}{2K^d} + \sum_{i=1}^{d-1} \frac{\beta_i}{K^i} : \mathbf{x} \in \{0, 1, \dots, K-1\}^d \right\} \\ &= \left\{ \frac{i}{K^{d-1}} + \frac{k}{2K^d} : i = 0, 1, \dots, K^{d-1}-1 \quad \text{and} \quad k = 0, 1, \dots, K-1 \right\} = \mathcal{A}_1. \end{aligned}$$

728 **Step 3.2:** Construct g to satisfy $g \circ \psi_1(\mathbf{x}) = \tilde{f}(\mathbf{x})$ and to meet the requirements of applying
729 Proposition B.2.

730 Let $g : [0, 1] \rightarrow \mathbb{R}$ be a continuous piecewise linear function with a set of breakpoints

731
$$\left\{ \frac{j}{2K^d} : j = 0, 1, \dots, 2K^d \right\} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{1\}.$$

732 Moreover, the values of g at these breakpoints are assigned as follows:

733 • At the breakpoint 1, let $g(1) = \tilde{f}(\mathbf{1})$, where $\mathbf{1} = [1, 1, \dots, 1]^T \in \mathbb{R}^d$.

734 • For the breakpoints in $\mathcal{A}_1 = \left\{ \psi_1(\mathbf{x}) : \mathbf{x} \in \{0, 1, \dots, K-1\}^d \right\}$, we set

735
$$g(\psi_1(\mathbf{x})) = \tilde{f}(\mathbf{x}) \quad \text{for any } \mathbf{x} \in \{0, 1, \dots, K-1\}^d. \quad (8)$$

736 • The values of g at the breakpoints in \mathcal{A}_2 are assigned to reduce the variation of g , which is a
737 requirement of applying Proposition B.2. Recall that

738
$$\left\{ \frac{i}{K^{d-1}} - \frac{K+1}{2K^d}, \frac{i}{K^{d-1}} \right\} \subseteq \mathcal{A}_1 \cup \{1\} \quad \text{for } i = 1, 2, \dots, K^{d-1},$$

739 implying the values of g at $\frac{i}{K^{d-1}} - \frac{K+1}{2K^d}$ and $\frac{i}{K^{d-1}}$ have been assigned in the previous
740 cases for. Thus, the values of g at the breakpoints in \mathcal{A}_2 can be successfully assigned
741 by letting g linear on each interval $[\frac{i}{K^{d-1}} - \frac{K+1}{2K^d}, \frac{i}{K^{d-1}}]$ for $i = 1, 2, \dots, K^{d-1}$ since $\mathcal{A}_2 \subseteq$
742 $\bigcup_{i=1}^{K^{d-1}} [\frac{i}{K^{d-1}} - \frac{K+1}{2K^d}, \frac{i}{K^{d-1}}]$. See Figure 14 for an illustration.

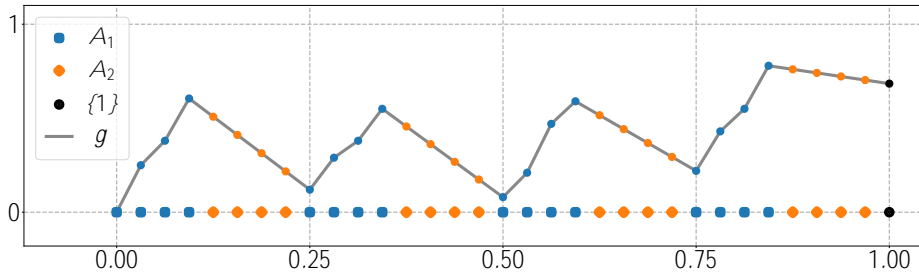


Figure 14: An illustration of \mathcal{A}_1 , \mathcal{A}_2 , $\{1\}$, and g for $K = 4$ and $d = 2$.

743 Apparently, such a function g exists. See Figure 14 for an illustration of g . It is easy to verify that

$$744 \quad \left| g\left(\frac{j}{2K^d}\right) - g\left(\frac{j-1}{2K^d}\right) \right| \leq \max \left\{ \omega_{\tilde{f}}\left(\frac{\sqrt{d}}{K}\right), \frac{\omega_{\tilde{f}}(\sqrt{d})}{K} \right\} \leq \omega_{\tilde{f}}\left(\frac{\sqrt{d}}{K}\right) = \omega_f\left(\frac{\sqrt{d}}{K}\right)$$

745 for $j = 1, 2, \dots, 2K^d$. Moreover, we have

$$746 \quad 0 \leq g\left(\frac{j}{2K^d}\right) \leq 2\omega_f(\sqrt{d}) \quad \text{for } j = 0, 1, \dots, 2K^d.$$

747 **Step 3.3:** Construct ψ_2 approximating g well on $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \{1\}$.

748 Observe that

$$749 \quad 2K^d = 2(\lfloor n^{(s+1)/d} \rfloor)^d \leq 2n^{s+1} \leq (2n)^{s+1} = \tilde{n}^{s+1}, \quad \text{where } \tilde{n} = 2n.$$

750 By Proposition B.2 with $y_j = g\left(\frac{j}{2K^d}\right)$ and $\varepsilon = \omega_f\left(\frac{\sqrt{d}}{K}\right) > 0$ therein, there exists

$$751 \quad \tilde{\psi}_2 \in \mathcal{NN}_s \left\{ 350(s+7)^2(\tilde{n}+1) \right\} = \mathcal{NN}_s \left\{ 350(s+7)^2(2n+1) \right\}$$

752 such that

$$753 \quad \left| \tilde{\psi}_2(j) - g\left(\frac{j}{2K^d}\right) \right| \leq \omega_f\left(\frac{\sqrt{d}}{K}\right) \quad \text{for } j = 0, 1, \dots, 2K^d - 1$$

754 and

$$755 \quad 0 \leq \tilde{\psi}_2(x) \leq \max \left\{ g\left(\frac{j}{2K^d}\right) : j = 0, 1, \dots, 2K^d - 1 \right\} \leq 2\omega_f(\sqrt{d}) \quad \text{for any } x \in \mathbb{R}.$$

756 By defining $\psi_2(x) := \tilde{\psi}_2(2K^d x)$ for any $x \in \mathbb{R}$, we have

$$757 \quad 0 \leq \psi_2(x) = \tilde{\psi}_2(2K^d x) \leq 2\omega_f(\sqrt{d}) \quad \text{for any } x \in \mathbb{R} \quad (9)$$

758 and

$$759 \quad \left| \psi_2\left(\frac{j}{2K^d}\right) - g\left(\frac{j}{2K^d}\right) \right| = \left| \tilde{\psi}_2(j) - g\left(\frac{j}{2K^d}\right) \right| \leq \omega_f\left(\frac{\sqrt{d}}{K}\right) \quad \text{for } j = 0, 1, \dots, 2K^d - 1. \quad (10)$$

760 Let us end Step 3 by defining the desired function ϕ_2 as $\phi_2 := \psi_2 \circ \psi_1$. Recall that $\psi_1(\cdot) = \mathcal{A}_1 \subseteq$
761 $\left\{ \frac{j}{2K^d} : j = 0, 1, \dots, 2K^d - 1 \right\}$. Then, by Equations (8) and (10), we have

$$762 \quad \left| \phi_2\left(\cdot\right) - \tilde{f}(\mathbf{x}) \right| = \left| \psi_2(\psi_1(\cdot)) - g(\psi_1(\cdot)) \right| \leq \omega_f\left(\frac{\sqrt{d}}{K}\right) \quad (11)$$

763 for any $\cdot \in \{0, 1, \dots, K-1\}^d$. Moreover, by Equation (9) and $\phi_2 = \psi_2 \circ \psi_1$, we have

$$764 \quad 0 \leq \phi_2(\mathbf{x}) = \psi_2(\psi_1(\mathbf{x})) \leq 2\omega_f(\sqrt{d}) \quad \text{for any } \mathbf{x} \in \mathbb{R}^d. \quad (12)$$

765 **Step 4:** Construct the final network to implement the desired function ϕ .

766 Define $\phi := \phi_2 \circ \Phi_1 + f(\mathbf{0}) - \omega_f(\sqrt{d})$. By Equation (12), we have

$$767 \quad 0 \leq \phi_2 \circ \Phi_1(\mathbf{x}) \leq 2\omega_f(\sqrt{d})$$

768 for any $\mathbf{x} \in \mathbb{R}^d$, implying

$$769 \quad f(\mathbf{0}) - \omega_f(\sqrt{d}) \leq \phi(\mathbf{x}) = \phi_2 \circ \Phi_1(\mathbf{x}) + f(\mathbf{0}) - \omega_f(\sqrt{d}) \leq f(\mathbf{0}) + \omega_f(\sqrt{d}).$$

770 It follows that $\|\phi\|_{L^\infty(\mathbb{R}^d)} \leq |f(\mathbf{0})| + \omega_f(\sqrt{d})$.

771 Next, let us estimate the approximation error. Recall that $f = \tilde{f} + f(\mathbf{0}) - \omega_f(\sqrt{d})$ and $\phi = \phi_2 \circ \Phi_1 +$
772 $f(\mathbf{0}) - \omega_f(\sqrt{d})$. By Equations (6) and (11), for any $\mathbf{x} \in Q$ and $\cdot \in \{0, 1, \dots, K-1\}^d$, we have

$$\begin{aligned} 773 \quad |f(\mathbf{x}) - \phi(\mathbf{x})| &= \left| \tilde{f}(\mathbf{x}) - \phi_2 \circ \Phi_1(\mathbf{x}) \right| = \left| \tilde{f}(\mathbf{x}) - \phi_2(\cdot) \right| \\ &\leq \left| \tilde{f}(\mathbf{x}) - \tilde{f}(\mathbf{x}) \right| + \left| \tilde{f}(\mathbf{x}) - \phi_2(\cdot) \right| \\ &\leq \omega_f\left(\frac{\sqrt{d}}{K}\right) + \omega_f\left(\frac{\sqrt{d}}{K}\right) \leq 2\omega_f\left(2\sqrt{d}n^{-(s+1)/d}\right), \end{aligned}$$

774 where the last inequality comes from the fact

$$775 \quad K = \lfloor n^{(s+1)/d} \rfloor \geq n^{(s+1)/d}/2 \quad \text{for } n \in \mathbb{N}^+.$$

776 Recall the fact $\omega_f(j \cdot r) \leq j \cdot \omega_f(r)$ for any $j \in \mathbb{N}^+$ and $r \in [0, \infty)$. Therefore, for any $\mathbf{x} \in$
 777 $\bigcup_{\sigma \in \{0,1,\dots,K-1\}^d} Q = [0, 1]^d \setminus \Omega([0, 1]^d, K, \delta)$, we have

$$778 \quad |\phi(\mathbf{x}) - f(\mathbf{x})| \leq 2\omega_f\left(2\sqrt{d}n^{-(s+1)/d}\right) \leq 2\lceil 2\sqrt{d} \rceil \omega_f\left(n^{-(s+1)/d}\right) \\ \leq 6\sqrt{d}\omega_f\left(n^{-(s+1)/d}\right).$$

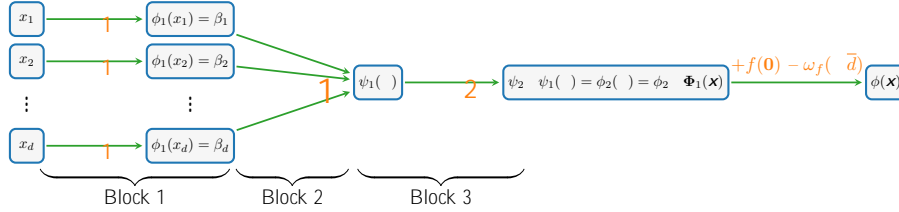


Figure 15: An illustration of the final NestNet realizing $\phi = \phi_2 \circ \Phi_1 + f(\mathbf{0}) - \omega_f(\sqrt{d})$ for $\mathbf{x} = [x_1, x_2, \dots, x_d]^T \in Q$ for each $\sigma \in \{0, 1, \dots, K-1\}^d$.

779 It remains to estimate the number of parameters in the NestNet realizing ϕ , which is shown in
 780 Figure 15. Recall that $\phi_1 \in \mathcal{NN}_s\{72(s+7)n\}$, ψ_1 is an affine linear map, and $\psi_2 \in \mathcal{NN}_s\{350(s+7)^2(2n+1)\}$. Therefore, $\phi = \phi_2 \circ \Phi_1 + f(\mathbf{0}) - \omega_f(\sqrt{d})$ can be realized by a height- s NestNet with
 781
 782 at most

$$783 \quad \underbrace{d^2(72(s+7)n)}_{\text{Block 1}} + \underbrace{(d+1)}_{\text{Block 2}} + \underbrace{350(s+7)^2(2n+1)}_{\text{Block 3}} + 1 \leq 355d^2(s+7)^2(2n+1)$$

784 parameters, which means we finish the proof of Theorem A.1.

785 C Proof of Proposition B.1

786 The key point of proving Proposition B.1 is the composition architecture of neural networks. To
 787 simplify the proof, we first establish several lemmas for proving Proposition B.1 in Section C.1. Next,
 788 we present the detailed proof of Proposition B.1 in Section C.2 based on the lemmas established in
 789 Section C.1.

790 C.1 Lemmas for proving Proposition B.1

791 **Lemma C.1.** Given any $n, r \in \mathbb{N}^+$ and $\delta \in (0, \frac{1}{C(r,n)})$ with $C(r,n) = \prod_{i=1}^r 2^{n^i}$, there exists
 792 $\phi \in \mathcal{NN}_r\{(12r+68)n\}$ such that

$$793 \quad \phi(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{2^{r^r}-1} [\ell, \ell+1 - C(r,n) \cdot \delta].$$

794 We will prove Lemma C.1 by induction. To simplify the proof, we introduce two lemmas for the base
 795 case and the induction step.

796 First, we introduce the following lemma for the base case of proving Lemma C.1.

797 **Lemma C.2.** Given any $n \in \mathbb{N}^+$ and $\delta \in (0, 1)$, there exists a function ϕ realized by a ReLU network
 798 of width 4 and depth $4n-1$ such that

$$799 \quad \phi(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{2^n-1} [\ell, \ell+1 - \delta].$$

800 *Proof.* Set $\tilde{\delta} = 2^{-n}\delta$ and define

$$801 \quad \phi_0(x) := \frac{\sigma(x-1+\tilde{\delta}) - \sigma(x-1)}{\tilde{\delta}} \quad \text{for } x \in \mathbb{R}.$$

802 Clearly, ϕ_0 can be realized by a one-hidden-layer ReLU network of width 2. Moreover, we have

$$803 \quad \phi_0(x) = \frac{\sigma(x-1+\tilde{\delta}) - \sigma(x-1)}{\tilde{\delta}} = \frac{0-0}{\tilde{\delta}} = 0 \quad \text{if } x \in [0, 1-\tilde{\delta}]$$

804 and

$$805 \quad \phi_0(x) = \frac{\sigma(x-1+\tilde{\delta}) - \sigma(x-1)}{\tilde{\delta}} = \frac{(x-1+\tilde{\delta}) - (x-1)}{\tilde{\delta}} = 1 \quad \text{if } x \in [1, 2-\tilde{\delta}].$$

806 By fixing

$$807 \quad x \in \bigcup_{\ell=0}^{2^n-1} [\ell, \ell+1-\delta] = \bigcup_{\ell=0}^{2^n-1} [\ell, \ell+1-2^n\tilde{\delta}],$$

808 we have $\lfloor x \rfloor \in \{0, 1, \dots, 2^n-1\}$, implying that $\lfloor x \rfloor$ can be represented as

$$809 \quad \lfloor x \rfloor = \sum_{i=0}^{n-1} z_i 2^i \quad \text{for } z_0, z_1, \dots, z_{n-1} \in \{0, 1\}.$$

810 Then, for $j = 0, 1, \dots, n-1$, we have $\sum_{i=0}^j z_i 2^i + 1 \leq z_j 2^j + \sum_{i=0}^{j-1} 2^i + 1 \leq z_j 2^j + 2^j$, implying

$$811 \quad \frac{x - \sum_{i=j+1}^{n-1} z_i 2^i}{2^j} \in \left[\frac{\lfloor x \rfloor - \sum_{i=j+1}^{n-1} z_i 2^i}{2^j}, \frac{\lfloor x \rfloor + 1 - 2^n \tilde{\delta} - \sum_{i=j+1}^{n-1} z_i 2^i}{2^j} \right] = \left[\frac{\sum_{i=0}^j z_i 2^i}{2^j}, \frac{\sum_{i=0}^j z_i 2^i + 1 - 2^n \tilde{\delta}}{2^j} \right]$$

$$\subseteq \left[\frac{z_j 2^j}{2^j}, \frac{z_j 2^j + 2^j - 2^n \tilde{\delta}}{2^j} \right] \subseteq [z_j, z_j + 1 - \tilde{\delta}].$$

812 It follows that

$$813 \quad \phi_0\left(\frac{x - \sum_{i=j+1}^{n-1} z_i 2^i}{2^j}\right) = z_j \quad \text{for } j = 0, 1, \dots, n-1.$$

814 Therefore, the desired function ϕ can be realized by the network in Figure 16.

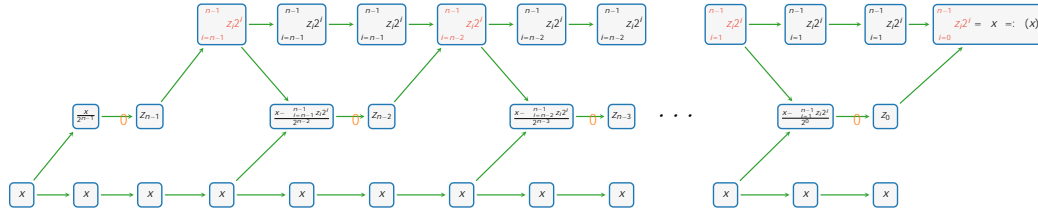


Figure 16: An illustration of the NestNet realizing ϕ . Here, ϕ_0 represent an one-hidden-layer ReLU network of width 2.

815 Clearly,

$$816 \quad \phi(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{2^n-1} [\ell, \ell+1-\delta].$$

817 Moreover, ϕ can be realized by a ReLU network of width $1+2+1=4$ and depth $(1+1+1)+(1+1+1)(n-1)=4n-1$. Hence, we finish the proof of Lemma C.2. \square

819 Next, we introduce the following lemma for the induction step of proving Lemma C.1.

820 **Lemma C.3.** Given any $n, s, \tilde{n} \in \mathbb{N}^+$ and $\delta \in (0, \frac{1}{2^{n^{s+1}}})$, if $g \in \mathcal{NN}_s\{\tilde{n}\}$ satisfying

$$821 \quad g(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{2^{\tilde{n}^s}-1} [\ell, \ell+1-\delta].$$

822 Then there exists $\phi \in \mathcal{NN}_{s+1}\{\tilde{n}+12n-7\}$ such that

$$823 \quad \phi(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{2^{n^{s+1}}-1} [\ell, \ell+1-2^{n^{s+1}}\delta].$$

824 *Proof.* By setting $m = 2^{n^s}$, we have $m^n = (2^{n^s})^n = 2^{(n^s)n} = 2^{n^{s+1}}$ and

$$825 \quad g(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{m-1} [\ell, \ell + 1 - \delta]. \quad (13)$$

826 By fixing

$$827 \quad x \in \bigcup_{\ell=0}^{2^{n^{s+1}}-1} [\ell, \ell + 1 - 2^{n^{s+1}} \delta] = \bigcup_{\ell=0}^{m^n-1} [\ell, \ell + 1 - m^n \delta],$$

828 we have $\lfloor x \rfloor \in \{0, 1, \dots, m^n - 1\}$, implying that $\lfloor x \rfloor$ can be represented as

$$829 \quad \lfloor x \rfloor = \sum_{i=0}^{n-1} z_i m^i \quad \text{for } z_0, z_1, \dots, z_{n-1} \in \{0, 1, \dots, m-1\}.$$

830 Then, for $j = 0, 1, \dots, n-1$, we have

$$831 \quad \sum_{i=0}^j z_i m^i + 1 \leq z_j m^j + \sum_{i=0}^{j-1} (m-1)m^i + 1 = z_j m^j + m^j,$$

832 implying

$$\begin{aligned} \frac{x - \sum_{i=j+1}^{n-1} z_i m^i}{m^j} &\in \left[\frac{\lfloor x \rfloor - \sum_{i=j+1}^{n-1} z_i m^i}{m^j}, \frac{\lfloor x \rfloor + 1 - m^n \delta - \sum_{i=j+1}^{n-1} z_i m^i}{m^j} \right] \\ 833 \quad &= \left[\frac{\sum_{i=0}^j z_i m^i}{m^j}, \frac{\sum_{i=0}^j z_i m^i + 1 - m^n \delta}{m^j} \right] \\ &\subseteq \left[\frac{z_j m^j}{m^j}, \frac{z_j m^j + m^j - m^n \delta}{m^j} \right] \subseteq [z_j, z_j + 1 - \delta]. \end{aligned}$$

834 It follows that

$$835 \quad g\left(\frac{x - \sum_{i=j+1}^{n-1} z_i m^i}{m^j}\right) = z_j \quad \text{for } j = 0, 1, \dots, n-1.$$

836 Therefore, the desired function ϕ can be realized by the network in Figure 17.

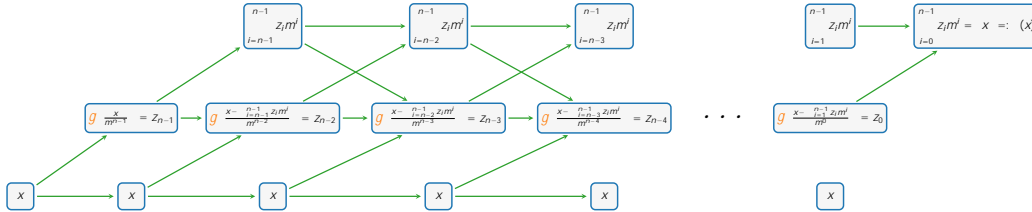


Figure 17: An illustration of the NestNet realizing ϕ . Here, g is regarded as an activation function.

837 Clearly,

$$838 \quad \phi(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{m^n-1} [\ell, \ell + 1 - m^n \delta] = \bigcup_{\ell=0}^{2^{n^{s+1}}-1} [\ell, \ell + 1 - 2^{n^{s+1}} \delta].$$

839 Moreover, the fact $g \in \mathcal{NN}_s\{\widehat{n}\}$ implies that ϕ can be realized by a height- $(s+1)$ NestNet with at most

$$841 \quad \underbrace{(1+1)2 + (2+1)3 + (3+1)3(n-2) + (3+1)}_{\text{outer network}} + \underbrace{\widehat{n}}_g = \widehat{n} + 12n - 7$$

842 parameters. Hence, we finish the proof of Lemma C.3. \square

843 With Lemmas C.2 and C.3 in hand, we are ready to prove Lemma C.1.

844 *Proof of Lemma C.1.* We will use the mathematical induction to prove Lemma C.1. First, we consider
845 the base case $r = 1$. By Lemma C.2, there exists a function ϕ realized by a ReLU network of width 4
846 and depth $4n - 1$ such that

$$847 \quad \phi(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{2^n-1} [\ell, \ell + 1 - \delta] \subseteq \bigcup_{\ell=0}^{2^n-1} [\ell, \ell + 1 - C(r, n) \cdot \delta] \text{ with } r = 1.$$

848 Moreover, the network realizing ϕ has at most $(4+1)4((4n-1)+1) = 80n$ parameters, implying
 849 $\phi \in \mathcal{NN}_1\{80n\} \subseteq \mathcal{NN}_1\{(12r+68)n\}$ for $r = 1$. Thus, the base case $r = 1$ is proved.

850 Next, assume Lemma C.1 holds for $r = s \in \mathbb{N}^+$. We need to show it is also true for $r = s + 1$. By the
 851 induction hypothesis, there exists $g \in \mathcal{NN}_s\{(12s+68)n\}$ such that

$$852 \quad g(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{2^{n^s}-1} [\ell, \ell + 1 - C(s, n) \cdot \delta].$$

853 By Lemma C.3 with $\widehat{n} = (12s+68)n$ therein and setting $\widehat{\delta} = C(s, n) \cdot \delta$, there exists

$$854 \quad \phi \in \mathcal{NN}_{s+1}\{\widehat{n} + 12n - 7\} \subseteq \mathcal{NN}_{s+1}\{(12s+68)n + 12n - 7\} \subseteq \mathcal{NN}_{s+1}\{(12(s+1)+68)n\}$$

855 such that

$$856 \quad \phi(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{2^{n^{s+1}}-1} [\ell, \ell + 1 - 2^{n^{s+1}}\widehat{\delta}].$$

857 Observe that

$$858 \quad 2^{n^{s+1}}\widehat{\delta} = 2^{n^{s+1}}C(s, n) \cdot \delta = 2^{n^{s+1}}\left(\prod_{i=1}^s 2^{n^i}\right) \cdot \delta = \left(\prod_{i=1}^{s+1} 2^{n^i}\right) \cdot \delta = C(s+1, n) \cdot \delta.$$

859 It follows that

$$860 \quad \phi(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{2^{n^{s+1}}-1} [\ell, \ell + 1 - C(s+1, n) \cdot \delta].$$

861 Thus, Lemma C.1 is proved for the case $r = s + 1$, which means we finish the induction step. Hence,
 862 by the principle of induction, we complete the proof of Lemma C.1. \square

863 C.2 Detailed proof of Proposition B.1

864 Set $C(r, n) = \prod_{i=1}^r 2^{n^i}$ and $\widetilde{\delta} = \frac{\delta}{C(r, n)} \in (0, \frac{1}{C(r, n)})$. By Lemma C.1, there exists $\phi_0 \in \mathcal{NN}_r\{(12r +$
 865 $68)n\}$ such that

$$866 \quad \phi_0(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{2^{n^r}-1} [\ell, \ell + 1 - C(r, n) \cdot \widetilde{\delta}] = \bigcup_{\ell=0}^{2^{n^r}-1} [\ell, \ell + 1 - \delta].$$

867 It follows from $J \leq 2^{n^r}$ that

$$868 \quad \phi_0(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{j=0}^{J-1} [j, j + 1 - \delta].$$

869 Set

$$870 \quad \widetilde{M} = \max_{x \in [J, J+1]} |\phi_0(x)| \quad \text{and} \quad M = \frac{\widetilde{M} + J}{\delta}.$$

871 Then, for any $x \in [J, J + 1]$, we have

$$872 \quad \phi_0(x) + M\sigma(x - (J - \delta)) \geq -\widetilde{M} + M\delta = -\widetilde{M} + (\widetilde{M} + J) = J,$$

873 implying

$$874 \quad \min \left\{ \phi_0(x) + M\sigma(x - (J - \delta)), J \right\} = J.$$

875 Moreover, for any $x \in \bigcup_{j=0}^{J-1} [j, j + 1 - \delta]$, we have $\sigma(x - (J - \delta)) = 0$, implying

$$876 \quad \min \left\{ \phi_0(x) + M\sigma(x - (J - \delta)), J \right\} = \min \left\{ \phi_0(x), J \right\} = \min \left\{ \lfloor x \rfloor, J \right\} = \lfloor x \rfloor.$$

877 Therefore, by defining

$$878 \quad \phi(x) := \min \left\{ \phi_0(x) + M\sigma(x - (J - \delta)), J \right\} \quad \text{for any } x \in \bigcup_{j=0}^J [j, j + 1 - \delta \cdot \mathbf{1}_{\{j \leq J-1\}}],$$

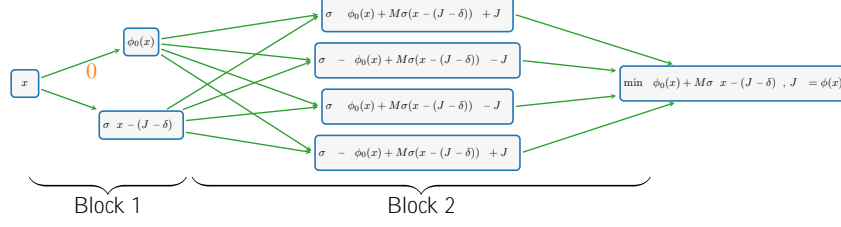


Figure 18: An illustration of the network realizing ϕ for any $x \in \bigcup_{j=0}^J [j, j + 1 - \delta \cdot \mathbf{1}_{\{j \leq J-1\}}]$ based on the fact $\min\{a, b\} = \frac{1}{2}(\sigma(a + b) - \sigma(-a - b) - \sigma(a - b) - \sigma(-a + b))$.

879 we have

$$880 \quad \phi(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{j=0}^{J-1} [j, j + 1 - \delta]$$

881 and

$$882 \quad \phi(x) = J \quad \text{for any } x \in [J, J + 1].$$

883 Moreover, ϕ can be realized by the network in Figure 18. The fact $\phi_0 \in \mathcal{NN}_r\{(12r + 68)n\}$ implies
884 that ϕ can be realized by a height- r NestNet with at most

$$885 \quad \underbrace{3((12r + 68)n)}_{\text{Block 1}} + \underbrace{(2 + 1)4 + (4 + 1)}_{\text{Block 2}} \leq 36(r + 7)n$$

886 parameters. So we finish the proof of Proposition B.1.

887 D Proof of Proposition B.2

888 The key idea of proving Proposition B.2 is the bit extraction technique proposed in [6]. First, we
889 establish several lemmas for proving Proposition B.2 and give their proofs in Section D.1 except for
890 Lemma D.2, the proof of which is placed in Section D.3 since it is complicated. Next, we present the
891 detailed proof of Proposition B.2 in Section D.2 based on the lemmas established in Section D.1.

892 D.1 Lemmas for proving Proposition B.2

893 To simplify the proof of Proposition B.2, we establish several lemmas as the intermediate step. We
894 first establish a lemma to show that any continuous piecewise linear functions on \mathbb{R} can be realized
895 by one-hidden-layer ReLU networks.

896 **Lemma D.1.** *Given any $p \in \mathbb{N}^+$, any continuous piecewise linear function on \mathbb{R} with at most p
897 breakpoints can be realized by a one-hidden-layer ReLU network of width $p + 1$.*

898 *Proof.* We will use the mathematical induction to prove Lemma D.1. First, we consider the base
899 case $p = 1$. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous piecewise linear function on \mathbb{R} with at most $p = 1$
900 breakpoints. Then there exist $a_1, a_2, x_0 \in \mathbb{R}$ such that

$$901 \quad f(x) = \begin{cases} a_1(x - x_0) + f(x_0) & \text{if } x \geq x_0 \\ a_2(x_0 - x) + f(x_0) & \text{if } x < x_0. \end{cases}$$

902 Thus, $f(x) = a_1\sigma(x - x_0) + a_2\sigma(x_0 - x) + f(x_0)$ for any $x \in \mathbb{R}$, implying f can be realized by a
903 one-hidden-layer ReLU network of width $2 = p + 1$ for $p = 1$. Hence, Lemma D.1 is proved for the
904 case $p = 1$.

905 Now, assume Lemma D.1 holds for $p = k \in \mathbb{N}^+$, we would like to show it is also true for $p = k + 1$.
906 Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous piecewise linear function on with at most $k + 1$ breakpoints. We
907 may assume the biggest breakpoint of f is x_0 since it is trivial for the case that f has no breakpoint.
908 Denote the slopes of the linear pieces left and right next to x_0 by a_1 and a_2 , respectively. Define

$$909 \quad \tilde{f}(x) := f(x) - (a_2 - a_1)\sigma(x - x_0) \quad \text{for any } x \in \mathbb{R}.$$

910 Then \tilde{f} has at most k breakpoints. By the induction hypothesis, \tilde{f} can be realized by a one-hidden-
 911 layer ReLU network of width $k + 1$. Thus, there exist $w_{0,j}, b_{0,j}, w_{1,j}, b_1$ for $j = 1, 2, \dots, k + 1$ such
 912 that

$$913 \quad \tilde{f}(x) = \sum_{j=1}^{k+1} w_{1,j} \sigma(w_{0,j}x + b_{0,j}) + b_1 \quad \text{for any } x \in \mathbb{R}.$$

914 Therefore, for any $x \in \mathbb{R}$, we have

$$915 \quad f(x) = (a_2 - a_1)\sigma(x - x_0) + \tilde{f}(x) = (a_2 - a_1)\sigma(x - x_0) + \sum_{j=1}^{k+1} w_{1,j} \sigma(w_{0,j}x + b_{0,j}) + b_1,$$

916 implying f can be realized by a one-hidden-layer ReLU network of width $k + 2 = (k + 1) + 1 = p + 1$
 917 for $p = k + 1$. Thus, we finish the induction process. Therefore, by the principle of induction, we
 918 complete the proof of Lemma D.1. \square

919 Next, we establish a lemma to extract the sum of n^s bits via a height- s NestNet with $\mathcal{O}(n)$ parameters.

920 **Lemma D.2.** *Given any $n, s \in \mathbb{N}^+$, there exists $\phi \in \mathcal{NN}_s\{57(s + 7)^2(n + 1)\}$ such that: For any
 921 $\theta_1, \theta_2, \dots, \theta_{n^s} \in \{0, 1\}$, we have*

$$922 \quad \phi(k + \text{bin}0.\theta_1\theta_2\cdots\theta_{n^s}) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 0, 1, \dots, n^s. \quad (14)$$

923 The proof of Lemma D.2 is complicated and hence is placed in Section D.3. Then, based on
 924 Lemma D.2, we establish a new lemma, Lemma D.3 below, which is a key intermediate conclusion
 925 to prove Proposition B.2.

926 **Lemma D.3.** *Given any $n, s \in \mathbb{N}^+$ and $\theta_{i,\ell} \in \{0, 1\}$ for $i = 0, 1, \dots, n - 1$ and $\ell = 0, 1, \dots, m - 1$, where
 927 $m = n^s$, there exists $\phi \in \mathcal{NN}_s\{58(s + 7)^2(n + 1)\}$ such that*

$$928 \quad \phi(j) = \sum_{\ell=0}^k \theta_{i,\ell} \quad \text{for } j = 0, 1, \dots, nm - 1,$$

929 where (i, k) is the unique index pair satisfying $j = im + k$ with $i \in \{0, 1, \dots, n - 1\}$ and $k \in$
 930 $\{0, 1, \dots, m - 1\}$.

931 *Proof.* We first construct a network to extract the unique index pair (i, k) from $j \in \{0, 1, \dots, nm - 1\}$
 932 with the following condition

$$933 \quad j = im + k \quad \text{with } i \in \{0, 1, \dots, n - 1\} \text{ and } k \in \{0, 1, \dots, m - 1\}.$$

934 There exists a continuous piecewise linear function ϕ_1 with $2n$ breakpoints such that

$$935 \quad \phi_1(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{n-1} [\ell, \ell + 1 - \delta] \text{ with } \delta = \frac{1}{2m}.$$

936 By Lemma D.1, ϕ_1 can be realized by a one-hidden-layer ReLU network of width $2n + 1$. Moreover,
 937 for any $j \in \{0, 1, \dots, nm - 1\}$, we have

$$938 \quad \phi_1\left(\frac{j}{m}\right) = \lfloor \frac{j}{m} \rfloor = i \quad \text{and} \quad j - m\phi_1\left(\frac{j}{m}\right) = j - mi = k,$$

939 where (i, k) is the unique index pair satisfying $j = im + k$ with $i \in \{0, 1, \dots, n - 1\}$ and $k \in$
 940 $\{0, 1, \dots, m - 1\}$. By defining

$$941 \quad \Phi_1(x) := \begin{bmatrix} \phi_1\left(\frac{x}{m}\right) \\ x - m\phi_1\left(\frac{x}{m}\right) \end{bmatrix} \quad \text{for any } x \geq 0,$$

942 we have

$$943 \quad \Phi_1(j) = \begin{bmatrix} \phi_1\left(\frac{j}{m}\right) \\ j - m\phi_1\left(\frac{j}{m}\right) \end{bmatrix} = \begin{bmatrix} i \\ k \end{bmatrix} \quad \text{for } j = 0, 1, \dots, nm - 1,$$

944 where (i, k) is the unique index pair satisfying $j = im + k$ with $i \in \{0, 1, \dots, n - 1\}$ and $k \in \{0, 1, \dots, m -$
 945 $1\}$. Moreover, Φ_1 can be realized by a one-hidden-layer ReLU network of width $2(2n + 1) + 1 = 4n + 3$.
 946 Hence, the network realizing Φ_1 has at most $(1 + 1)(4n + 3) + ((4n + 3) + 1)2 = 16n + 14$ parameters.

947 Define

$$948 \quad z_i := \text{bin}0.\theta_{i,0}\theta_{i,1}\cdots\theta_{i,m-1} \quad \text{for } i = 0, 1, \dots, n-1.$$

949 There exists a continuous piecewise linear function $\tilde{\phi}_2$ with n breakpoints such that

$$950 \quad \tilde{\phi}_2(i) = z_i \quad \text{for } i = 0, 1, \dots, n-1.$$

951 By Lemma D.1, $\tilde{\phi}_2$ can be realized by a one-hidden-layer ReLU network of width $n+1$.

952 By Lemma D.2, there exists $\phi_3 \in \mathcal{NN}_s\{57(s+7)^2(n+1)\}$ such that: For any $\xi_1, \xi_2, \dots, \xi_{n^s} \in \{0, 1\}$,
953 we have

$$954 \quad \phi_3(k + \text{bin}0.\xi_1\xi_2\cdots\xi_{n^s}) = \sum_{\ell=1}^k \xi_\ell \quad \text{for } k = 1, 2, \dots, n^s.$$

955 It follows from $m = n^s$ that, for any $\xi_0, \xi_1, \dots, \xi_{m-1} \in \{0, 1\}$, we have

$$956 \quad \phi_3(k + \text{bin}0.\xi_0\xi_1\cdots\xi_{m-1}) = \sum_{\ell=1}^k \xi_{\ell-1} = \sum_{\ell=0}^{k-1} \xi_\ell \quad \text{for } k = 1, 2, \dots, m,$$

957 implying

$$958 \quad \phi_3(k+1 + \text{bin}0.\xi_0\xi_1\cdots\xi_{m-1}) = \sum_{\ell=0}^k \xi_\ell \quad \text{for } k = 0, 1, \dots, m-1.$$

959 Then, for $i = 0, 1, \dots, n-1$ and $k = 0, 1, \dots, m-1$, we have

$$960 \quad \phi_3(k+1 + \tilde{\phi}_2(i)) = \phi_2(k+1 + z_i) = \phi_3(k+1 + \text{bin}0.\theta_{i,0}\theta_{i,1}\cdots\theta_{i,m-1}) = \sum_{\ell=0}^k \theta_{i,\ell}.$$

961 By defining

$$962 \quad \phi_2(x, y) := y + 1 + \tilde{\phi}_2(x) \quad \text{for any } x, y \in [0, \infty)$$

963 and $\phi := \phi_3 \circ \phi_2 \circ \Phi_1$, we have

$$964 \quad \phi(j) = \phi_3 \circ \phi_2 \circ \Phi_1(j) = \phi_3 \circ \phi_2(i, k) = \phi_3(k+1 + \tilde{\phi}_2(i)) = \sum_{\ell=0}^k \theta_{i,\ell}$$

965 for $j = 0, 1, \dots, nm-1$, where (i, k) is the unique index pair satisfying $j = im + k$ with $i \in$
966 $\{0, 1, \dots, n-1\}$ and $k \in \{0, 1, \dots, m-1\}$.

967 It remains to estimate the number of parameters in the NestNet realizing $\phi = \phi_3 \circ \phi_2 \circ \Phi_1$. Observe
968 that ϕ_2 can be realized by a one-hidden-layer ReLU network of width $(n+1) + 1 = n+2$. Then, the
969 network realizing ϕ_2 has at most $(2+1)(n+2) + ((n+2)+1) = 4n+9$ parameters. Therefore, ϕ
970 can be realized by a height- s NestNet with at most

$$971 \quad \underbrace{(16n+14)}_{\Phi_1} + \underbrace{(4n+9)}_{\phi_2} + \underbrace{57(s+7)^2(n+1)}_{\phi_3} \leq 58(s+7)^2(n+1)$$

972 parameters, which means we complete the proof of Lemma D.3. \square

973 D.2 Detailed proof of Proposition B.2

974 We may assume $J = mn = n^{s+1}$ with $m = n^s$ since we can set $y_{J-1} = y_J = \dots = y_{mn-1}$ if $J < mn$.

975 Define

$$976 \quad a_j := \lfloor y_j/\varepsilon \rfloor \quad \text{for } j = 0, 1, \dots, nm-1.$$

977 Our goal is to construct a function ϕ such that $\phi(j) = a_j\varepsilon$ for $j = 0, 1, \dots, nm-1$.

978 For $i = 0, 1, \dots, n-1$, we define

$$979 \quad b_{i,\ell} = \begin{cases} 0 & \text{for } \ell = 0 \\ a_{im+\ell} - a_{im+\ell-1} & \text{for } \ell = 1, 2, \dots, m-1. \end{cases}$$

980 Since $|y_j - y_{j-1}| \leq \varepsilon$ for all j , we have $|a_j - a_{j-1}| \leq 1$. It follows that $b_{i,\ell} \in \{-1, 0, 1\}$ for $i =$
981 $0, 1, \dots, n-1$ and $\ell = 0, 1, \dots, m-1$. Hence, there exist $c_{i,\ell} \in \{0, 1\}$ and $d_{i,\ell} \in \{0, 1\}$ such that

$$982 \quad b_{i,\ell} = c_{i,\ell} - d_{i,\ell} \quad \text{for } i = 0, 1, \dots, n-1 \text{ and } \ell = 0, 1, \dots, m-1.$$

983 Since any $j \in \{0, 1, \dots, nm - 1\}$ can be uniquely indexed as $j = im + k$ with $i \in \{0, 1, \dots, n - 1\}$ and
 984 $k \in \{0, 1, \dots, m - 1\}$, we have

$$\begin{aligned} a_j &= a_{im+k} = a_{im} + \sum_{\ell=1}^k (a_{im+\ell} - a_{im+\ell-1}) = a_{im} + \sum_{\ell=1}^k b_{i,\ell} = a_{im} + \sum_{\ell=0}^k b_{i,\ell} \\ &= a_{im} + \sum_{\ell=0}^k c_{i,\ell} - \sum_{\ell=0}^k d_{i,\ell}. \end{aligned}$$

986 There exists a continuous piecewise linear function ϕ_1 with $2n$ breakpoints such that

$$\phi_1(x) = a_{im} \quad \text{for any } x \in [im, im + m - 1] \text{ and } i = 0, 1, \dots, n - 1.$$

988 Then, we have

$$\phi_1(j) = a_{im} \quad \text{for } j = 0, 1, \dots, nm - 1,$$

990 where (i, k) is the unique index pair satisfying $j = im + k$ with $i \in \{0, 1, \dots, n - 1\}$ and $k \in$
 991 $\{0, 1, \dots, m - 1\}$. By Lemma D.1, ϕ_1 can be realized by a one-hidden-layer ReLU network of width
 992 $2n + 1$.

993 By Lemma D.3, there exist $\phi_2, \phi_3 \in \mathcal{NN}_s\{58(s + 7)^2(n + 1)\}$ such that

$$\phi_2(j) = \sum_{\ell=0}^k c_{i,\ell} \quad \text{and} \quad \phi_3(j) = \sum_{\ell=0}^k d_{i,\ell} \quad \text{for } j = 0, 1, \dots, nm - 1,$$

995 where (i, k) is the unique index pair satisfying $j = im + k$ with $i \in \{0, 1, \dots, n - 1\}$ and $k \in$
 996 $\{0, 1, \dots, m - 1\}$.

997 Hence, by indexing $j \in \{0, 1, \dots, nm - 1\}$ as $j = im + k$ for $i \in \{0, 1, \dots, n - 1\}$ and $k \in \{0, 1, \dots, m - 1\}$,
 998 we have

$$a_j = a_{im} + \sum_{\ell=0}^k c_{i,\ell} - \sum_{\ell=0}^k d_{i,\ell} = \phi_1(j) + \phi_2(j) - \phi_3(j).$$

1000 By defining

$$\tilde{\phi}(x) := (\phi_1(x) + \phi_2(x) + \phi_3(x))\varepsilon \quad \text{for any } x \in \mathbb{R},$$

1002 we have $\tilde{\phi}(j) = a_j\varepsilon$ for $j = 0, 1, \dots, nm - 1$ and $\tilde{\phi}$ can be realized by the height- s NestNet in Figure 19.

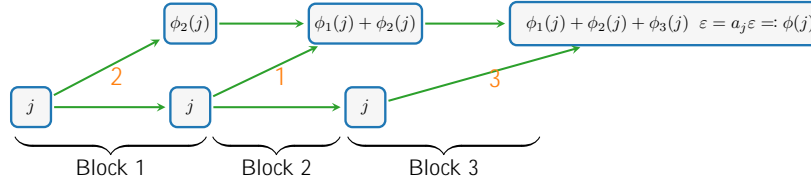


Figure 19: An illustration of the NestNet realizing $\tilde{\phi}$ for $j = 0, 1, \dots, J - 1$.

1003 In Figure 19, Block 1 or 3 has at most

$$3(58(s + 7)^2(n + 1)) = 174(s + 7)^2(n + 1)$$

1005 parameters; Block 2 is of width $(2n + 1) + 2 = 2n + 3$ and depth 1, and hence has at most

$$(2 + 1)(2n + 3) + ((2n + 3) + 1)2 = 10n + 17$$

1007 parameters. Then, $\tilde{\phi}$ can be realized by a height- s ReLU NestNet with at most

$$2(174(s + 7)^2(n + 1)) + 10n + 17 = 349(s + 7)^2(n + 1)$$

1009 parameters. Note that $\tilde{\phi}$ may not be bounded. Thus, we define

$$\psi(x) := \min\{\sigma(x), M\} \quad \text{for any } x \in \mathbb{R},$$

1011 where

1012
$$M = \max\{y_j : j = 0, 1, \dots, nm - 1\}.$$

1013 Then, the desired function ϕ can be define via $\phi := \psi \circ \tilde{\phi}$. Clearly,

1014
$$0 \leq \phi(x) \leq M = \max\{y_j : j = 0, 1, \dots, J - 1\} \quad \text{for any } x \in \mathbb{R}.$$

1015 It follows from $0 \leq a_j\varepsilon = \lfloor y_j/\varepsilon \rfloor \varepsilon \leq y_j \leq M$ for $j = 0, 1, \dots, J - 1$ that

1016
$$\phi(j) = \psi \circ \tilde{\phi}(j) = \psi(a_j\varepsilon) = \min\{\sigma(a_j\varepsilon), M\} = a_j\varepsilon,$$

1017 implying

1018
$$|\phi(j) - y_j| = |a_j\varepsilon - y_j| = \left| \lfloor y_j/\varepsilon \rfloor \varepsilon - y_j \right| = \left| \lfloor y_j/\varepsilon \rfloor - y_j/\varepsilon \right| \varepsilon \leq \varepsilon.$$

1019 It remains to show that ϕ can be realized by a height- s ReLU NestNet with the desired size. Clearly,
 1020 ψ can be realized by the network in Figure 20, which is of width 4 and depth 2.

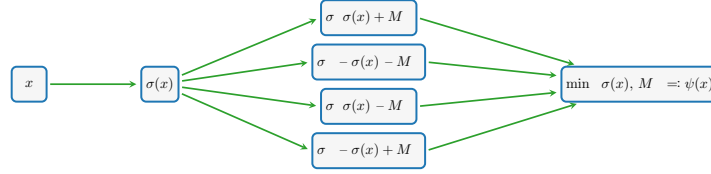


Figure 20: An illustration of the network realizing ψ based on the fact $\min\{a, b\} = \frac{1}{2}(\sigma(a + b) - \sigma(-a - b) - \sigma(a - b) - \sigma(-a + b))$.

1021 Therefore, ϕ can be realized by a height- s ReLU NestNet with at most

1022
$$349(s + 7)^2(n + 1) + (4 + 1)4(2 + 1) \leq 350(s + 7)^2(n + 1)$$

1023 parameters. Hence, we finish the proof of Proposition B.2.

1024 **D.3 Proof of Lemma D.2 for Proposition B.2**

1025 We will use the mathematical induction to prove Lemma D.2. To this end, we introduce two lemmas
 1026 for the base case and the induction step.

1027 **Lemma D.4.** *Given any $n \in \mathbb{N}^+$, there exists a function ϕ realized by a ReLU network with $128n + 294$
 1028 parameters such that: For any $\theta_1, \theta_2, \dots, \theta_n \in \{0, 1\}$, we have*

1029
$$\phi(k + \text{bin}0.\theta_1\theta_2\cdots\theta_n) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 0, 1, \dots, n. \quad (15)$$

1030 **Lemma D.5.** *Given any $n, r, \widehat{n} \in \mathbb{N}^+$, if $g \in \mathcal{NN}_r\{\widehat{n}\}$ satisfying*

1031
$$g(p + \text{bin}0.\xi_1\xi_2\cdots\xi_{n^r}) = \sum_{j=1}^p \xi_j \quad \text{for any } \xi_1, \xi_2, \dots, \xi_{n^r} \in \{0, 1\} \text{ and } p = 0, 1, \dots, n^r, \quad (16)$$

1032 *then there exists $\phi \in \mathcal{NN}_{r+1}\{\widehat{n} + 114(r + 7)(n + 1)\}$ such that: For any $\theta_1, \theta_2, \dots, \theta_{n^{r+1}} \in \{0, 1\}$, we
 1033 have*

1034
$$\phi(k + \text{bin}0.\theta_1\theta_2\cdots\theta_{n^{r+1}}) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 0, 1, \dots, n^{r+1}.$$

1035 The proofs of Lemmas D.4 and D.5 can be found in Sections D.3.1 and D.3.2, respectively. We
 1036 remark that the function ϕ in Lemma D.5 is independent of $\theta_1, \theta_2, \dots, \theta_{nm}$. The proof of Lemma D.2
 1037 mainly relies on Lemma D.4 and repeated applications of Lemma D.5. The details can be found
 1038 below.

1039 *Proof of Lemma D.2.* We will use the mathematical induction to prove Lemma D.2. First, let us
 1040 consider the base case $s = 1$. By Lemma D.4, there exists a function realized by a ReLU network
 1041 with $128n + 294$ parameters such that: For any $\theta_1, \theta_2, \dots, \theta_n \in \{0, 1\}$, we have

$$1042 \quad \phi(k + \text{bin}0.\theta_1\theta_2\cdots\theta_n) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 0, 1, \dots, n.$$

1043 That means Equation (14) holds for $s = 1$. Moreover, ϕ can also be regarded as a height-1 ReLU
 1044 NestNet with $128n + 294 \leq 57(s + 7)^2(n + 1)$ parameters for $s = 1$, which means Lemma D.2 is
 1045 proved for the case $s = 1$.

1046 Next, assume Lemma D.2 holds for $s = r \in \mathbb{N}^+$. We need to show that it is also true for $s = r + 1$ by
 1047 applying Lemma D.5. By the induction hypothesis, there exists

$$1048 \quad g \in \mathcal{NN}_r\{57(r + 7)^2(n + 1)\}$$

1049 such that: For any $\xi_1, \xi_2, \dots, \xi_{n^r} \in \{0, 1\}$, we have

$$1050 \quad g(k + \text{bin}0.\xi_1\xi_2\cdots\xi_{n^r}) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 0, 1, \dots, n^r.$$

1051 It follows from $m = n^r$ that

$$1052 \quad g(p + \text{bin}0.\xi_1\xi_2\cdots\xi_m) = \sum_{j=1}^p \xi_j \quad \text{for any } \xi_1, \xi_2, \dots, \xi_m \in \{0, 1\} \text{ and } p = 0, 1, \dots, m,$$

1053 which means g satisfies Equation (16). Then, by Lemma D.5 with $m = n^r$ and $\widehat{n} = 57(r + 7)^2(n + 1)$
 1054 therein, there exists

$$1055 \quad \phi \in \mathcal{NN}_{r+1}\{\widehat{n} + 114(r + 7)(n + 1)\}$$

1056 such that: For any $\theta_1, \theta_2, \dots, \theta_{nm} \in \{0, 1\}$, we have

$$1057 \quad \phi(k + \text{bin}0.\theta_1\theta_2\cdots\theta_{nm}) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 0, 1, \dots, nm.$$

1058 It follows from $m = n^r$ that, for any $\theta_1, \theta_2, \dots, \theta_{n^{r+1}} \in \{0, 1\}$, we have

$$1059 \quad \phi(k + \text{bin}0.\theta_1\theta_2\cdots\theta_{n^{r+1}}) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 0, 1, \dots, n^{r+1},$$

1060 which means Equation (14) holds for $s = r + 1$. Moreover, we have

$$\begin{aligned} 1061 \quad \widehat{n} + 114(r + 7)(n + 1) &= 57(r + 7)^2(n + 1) + 114(r + 7)(n + 1) \\ &= 57(n + 1)\left((r + 7)^2 + 2(r + 7)\right) \\ &\leq 57(n + 1)\left((r + 7) + 1\right)^2 = 57((r + 1) + 7)^2(n + 1). \end{aligned}$$

1062 This implies that

$$1063 \quad \phi \in \mathcal{NN}_{r+1}\{\widehat{n} + 114(r + 7)(n + 1)\} \subseteq \mathcal{NN}_{r+1}\{57((r + 1) + 7)^2(n + 1)\}.$$

1064 Thus, we prove Lemma D.2 for the case $s = r + 1$, which means we finish the induction step. Hence,
 1065 by the principle of induction, we complete the proof of Lemma D.2. \square

1066 D.3.1 Proof of Lemma D.4 for Lemma D.2

1067 To simplify the proof of Lemma D.4, we introduce the following lemma.

1068 **Lemma D.6.** *Given any $n \in \mathbb{N}^+$, there exists a function ϕ realized by a ReLU network of width 7 and*
 1069 *depth $2n + 1$ such that: For any $\theta_1, \theta_2, \dots, \theta_n \in \{0, 1\}$, we have*

$$1070 \quad \phi(\text{bin}0.\theta_1\theta_2\cdots\theta_n, k) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 0, 1, \dots, n.$$

1071 Lemma D.6 is the Lemma 3.5 of [35]. The detailed proof can be found therein. With Lemma D.6 in
 1072 hand, we are ready to prove Lemma D.4.

1073 *Proof of Lemma D.4.* By Lemma D.6, there exists a function ϕ_0 realized by a ReLU network of
 1074 width 7 and depth $2n + 1$ such that: For any $\theta_1, \theta_2, \dots, \theta_n \in \{0, 1\}$, we have

$$1075 \quad \phi_0(\text{bin}0.\theta_1\theta_2\cdots\theta_n, k) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 1, 2, \dots, n.$$

1076 The equation above is not true for $k = 0$. We will construct ϕ_2 such that

$$1077 \quad \phi_2(\text{bin}0.\theta_1\theta_2\cdots\theta_n, k) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 0, 1, \dots, n.$$

1078 To this end, we first set

$$1079 \quad M = \max \{|\phi_0(x, y)| : x \in [0, 1], y \in [0, n]\}$$

1080 and define

$$1081 \quad \phi_1(x, y) := \min \{M + \phi_0(x, y), 2My\} \quad \text{for any } x \in [0, 1] \text{ and } y \in [0, n].$$

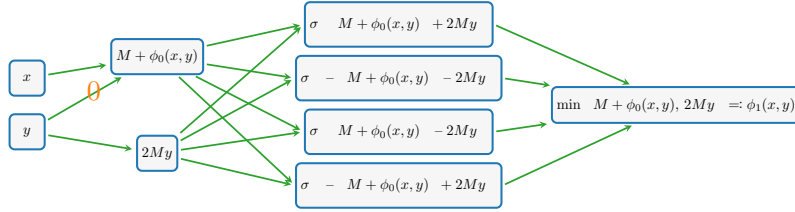


Figure 21: An illustration of the network realizing ϕ_1 for any $x \in [0, 1]$ and $y \in [0, n]$ based on the fact $\min\{a, b\} = \frac{1}{2}(\sigma(a + b) - \sigma(-a - b) - \sigma(a - b) - \sigma(-a + b))$.

1082 As we can see from Figure 21, ϕ_1 can be realized by a ReLU network of width $\max\{7, 4\} = 7$ and
 1083 depth $(2n + 1) + 2 = 2n + 3$. Moreover, we have

$$1084 \quad \begin{aligned} \phi_1(\text{bin}0.\theta_1\theta_2\cdots\theta_n, k) &= \min \{M + \phi_0(\text{bin}0.\theta_1\theta_2\cdots\theta_n, k), 2Mk\} \\ &= \begin{cases} M + \sum_{\ell=1}^k \theta_\ell & \text{for } k = 1, 2, \dots, n \\ 0 & \text{for } k = 0. \end{cases} \end{aligned}$$

1085 Define

$$1086 \quad \phi_2(x, y) := \sigma(\phi_1(x, y) - M) \quad \text{for any } x \in [0, 1] \text{ and } y \in [0, \infty).$$

1087 Then, ϕ_2 can be realized by a ReLU network of width 7 and depth $(2n + 3) + 1 = 2n + 4$. Moreover,
 1088 we have

$$1089 \quad \begin{aligned} \phi_2(\text{bin}0.\theta_1\theta_2\cdots\theta_n, k) &= \sigma(\phi_1(\text{bin}0.\theta_1\theta_2\cdots\theta_n, k) - M) \\ &= \begin{cases} \sigma(\sum_{\ell=1}^k \theta_\ell) = \sum_{\ell=1}^k \theta_\ell & \text{for } k = 1, 2, \dots, n \\ \sigma(-M) = 0 & \text{for } k = 0. \end{cases} \end{aligned}$$

1090 That is,

$$1091 \quad \phi_2(\text{bin}0.\theta_1\theta_2\cdots\theta_n, k) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 0, 1, \dots, n.$$

1092 Next, we will construct Ψ to extract k and $\text{bin}0.\theta_1\theta_2\cdots\theta_n$ from $k + \text{bin}0.\theta_1\theta_2\cdots\theta_n$. It is easy to
 1093 construct a continuous piecewise linear function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ with $2n$ breakpoints satisfying

$$1094 \quad \psi(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{n-1} [\ell, \ell + 1 - \delta] \text{ with } \delta = 2^{-n}.$$

1095 By Lemma D.1 with $p = 2n$ therein, ψ can be realized by a one-hidden-layer ReLU network of width
 1096 $2n + 1$. By defining

$$1097 \quad \Psi(x) := \begin{bmatrix} x - \psi(x) \\ \psi(x) \end{bmatrix} = \begin{bmatrix} \sigma(x) - \psi(x) \\ \psi(x) \end{bmatrix} \quad \text{for any } x \in [0, \infty).$$

1098 Then, Ψ can be realized by a one-hidden-layer ReLU network of width $1 + 2(2n + 1) = 4n + 3$. That
 1099 means, the network realizing Ψ has at most

$$1100 \quad (1 + 1)(4n + 3) + ((4n + 3) + 1)2 = 16n + 14$$

1101 parameters. Moreover, for any $\theta_1, \theta_2, \dots, \theta_n \in \{0, 1\}$ and $k = 0, 1, \dots, n$, we have

$$1102 \quad \psi(k + \text{bin}0.\theta_1\theta_2\cdots\theta_n) = \lfloor k + \text{bin}0.\theta_1\theta_2\cdots\theta_n \rfloor = k,$$

1103 implying

$$1104 \quad \Psi(k + \text{bin}0.\theta_1\theta_2\cdots\theta_n) = \begin{bmatrix} k + \text{bin}0.\theta_1\theta_2\cdots\theta_n - \psi(k + \text{bin}0.\theta_1\theta_2\cdots\theta_n) \\ \psi(k + \text{bin}0.\theta_1\theta_2\cdots\theta_n) \end{bmatrix} \\ = \begin{bmatrix} \text{bin}0.\theta_1\theta_2\cdots\theta_n \\ k \end{bmatrix}.$$

1105 Finally, the desired function ϕ can be defined via $\phi := \phi_2 \circ \Psi$. Clearly, the network realizing ϕ_2 is of
 1106 width 7 and depth $2n + 4$, and hence has at most

$$1107 \quad (7 + 1)7((2n + 4) + 1) = 56(2n + 5)$$

1108 parameters, implying ϕ can be realized by a ReLU network with at most

$$1109 \quad 56(2n + 5) + (16n + 14) = 128n + 294$$

1110 parameters. Moreover, for any $\theta_1, \theta_2, \dots, \theta_n \in \{0, 1\}$ and $k = 0, 1, \dots, n$, we have

$$1111 \quad \phi(k + \text{bin}0.\theta_1\theta_2\cdots\theta_n) = \phi_2 \circ \Psi(k + \text{bin}0.\theta_1\theta_2\cdots\theta_n) \\ = \phi_2(\text{bin}0.\theta_1\theta_2\cdots\theta_n, k) = \sum_{\ell=1}^k \theta_\ell.$$

1112 Thus, we finish the proof of Lemma D.4. □

1113 D.3.2 Proof of Lemma D.5 for Lemma D.2

1114 The key idea of proving Lemma D.5 is to construct a network with n blocks, each of which extracts
 1115 the sum of n^r bits via g . Then the whole network can extract the sum of n^{r+1} bits as we expect.

1116 To simplify our notation, we set $m = n^r$. Given any nm binary bits $\theta_\ell \in \{0, 1\}$ for $\ell = 1, 2, \dots, nm$,
 1117 we divide these nm bits into n classes according to their indices, where the i -th class is composed
 1118 of m bits $\theta_{im+1}, \dots, \theta_{i(m+m)}$ for $i = 0, 1, \dots, n - 1$. We will show how to extract the m bits of the i -th
 1119 class, stored in $\text{bin}0.\theta_{im+1}\cdots\theta_{i(m+m)}$.

1120 First, let us show how to construct a network to extract k and $\text{bin}0.\theta_1\theta_2\cdots\theta_{nm}$ from $k + 0.\theta_1\theta_2\cdots\theta_{nm}$.
 1121 By setting $\tilde{n} = 2n$ and Proposition B.1 with $J = 2^{\tilde{n}^r}$ therein, there exists

$$1122 \quad \tilde{g} \in \mathcal{NN}_r\{36(r + 7)\tilde{n}\} = \mathcal{NN}_r\{36(r + 7)(2n)\} = \mathcal{NN}_r\{72(r + 7)n\}$$

1123 such that

$$1124 \quad \tilde{g}(x) = \lfloor x \rfloor \quad \text{for any } x \in \bigcup_{\ell=0}^{J-1} [\ell, \ell + 1 - \delta].$$

1125 Observe that

$$1126 \quad J - 1 = 2^{\tilde{n}^r} - 1 = 2^{(2n)^r} - 1 \geq 2^{2(n^r)} - 1 = 2^{2m} - 1 = 4^m - 1 \geq m^2 \geq nm.$$

1127 It follows from $\text{bin}0.\theta_1\theta_2\cdots\theta_{nm} \leq 1 - 2^{-nm} = 1 - \delta$ that

$$1128 \quad k + \text{bin}0.\theta_1\theta_2\cdots\theta_{nm} \in \bigcup_{\ell=0}^{nm} [\ell, \ell + 1 - \delta] \subseteq \bigcup_{\ell=0}^{J-1} [\ell, \ell + 1 - \delta]$$

1129 for $k = 0, 1, \dots, nm$. Thus, we have

$$1130 \quad \tilde{g}(k + \text{bin}0.\theta_1\theta_2\cdots\theta_{nm}) = k \quad \text{for } k = 0, 1, \dots, nm. \quad (17)$$

1131 It is easy to verify that

$$1132 \quad 2^m \cdot \text{bin}0.\theta_{im+1}\cdots\theta_{nm} \in \bigcup_{\ell=0}^{2^m-1} [\ell, \ell + 1 - \delta] \quad \text{for } i = 0, 1, \dots, n-1.$$

1133 Since $2^m - 1 = 2^{n^r} - 1 \leq 2^{(2n)^r} - 1 = J - 1$, we have

$$1134 \quad \tilde{g}(2^m \cdot \text{bin}0.\theta_{im+1}\cdots\theta_{nm}) = \lfloor 2^m \cdot \text{bin}0.\theta_{im+1}\cdots\theta_{nm} \rfloor \quad \text{for } i = 0, 1, \dots, n-1.$$

1135 Therefore, for $i = 0, 1, \dots, n-1$, we have

$$1136 \quad \text{bin}0.\theta_{im+1}\cdots\theta_{im+m} = \frac{\lfloor 2^m \cdot \text{bin}0.\theta_{im+1}\cdots\theta_{nm} \rfloor}{2^m} = \frac{\tilde{g}(2^m \cdot \text{bin}0.\theta_{im+1}\cdots\theta_{nm})}{2^m}$$

1137 and

$$1138 \quad \begin{aligned} \text{bin}0.\theta_{(i+1)m+1}\cdots\theta_{nm} &= 2^m \left(\text{bin}0.\theta_{im+1}\cdots\theta_{nm} - \text{bin}0.\theta_{im+1}\cdots\theta_{im+m} \right) \\ &= 2^m \left(\text{bin}0.\theta_{im+1}\cdots\theta_{nm} - \frac{\tilde{g}(2^m \cdot \text{bin}0.\theta_{im+1}\cdots\theta_{nm})}{2^m} \right). \end{aligned}$$

1139 By defining

$$1140 \quad \phi_1(x) := \frac{\tilde{g}(2^m x)}{2^m} \quad \text{and} \quad \phi_2(x) := 2^m \left(x - \frac{\tilde{g}(2^m x)}{2^m} \right) = \left(\sigma(x) - \frac{\tilde{g}(2^m x)}{2^m} \right) \quad \text{for } x \geq 0,$$

1141 we have

$$1142 \quad \text{bin}0.\theta_{im+1}\cdots\theta_{im+m} = \phi_1(\text{bin}0.\theta_{im+1}\cdots\theta_{nm}) \quad (18)$$

1143 and

$$1144 \quad \text{bin}0.\theta_{(i+1)m+1}\cdots\theta_{nm} = \phi_2(\text{bin}0.\theta_{im+1}\cdots\theta_{nm}) \quad (19)$$

1145 for any $i \in \{0, 1, \dots, n-1\}$. Moreover, ϕ_1 can be realized by a one-hidden-layer \tilde{g} -activated network
1146 of width 1; ϕ_2 can be realized by a one-hidden-layer (σ, \tilde{g}) -activated network of width 2.

1147 Define

$$1148 \quad \phi_{3,i}(x) := \min\{\sigma(x - im), m\} \quad \text{for any } x \in \mathbb{R} \text{ and } i = 0, 1, \dots, n-1.$$

1149 For any $k \in \{1, 2, \dots, nm\}$, there exist $k_1 \in \{0, 1, \dots, n-1\}$ and $k_2 \in \{1, 2, \dots, m\}$ such that $k =$
1150 $k_1 m + k_2$. Then we have

$$1151 \quad \phi_{3,i}(k) = \min\{\sigma(k - im), m\} = \begin{cases} m & \text{if } i \leq k_1 - 1 \\ k_2 & \text{if } i = k_1 \\ 0 & \text{if } i \geq k_1 + 1. \end{cases} \quad (20)$$

1152 Observe that

$$1153 \quad \begin{aligned} \{1, 2, \dots, k\} &= \{1, 2, \dots, k_1 m + k_2\} \\ &= \left(\bigcup_{i=1}^{k_1-1} \{im + j : j = 1, 2, \dots, m\} \right) \cup \{k_1 m + j : j = 1, 2, \dots, k_2\}. \end{aligned}$$

1154 It follows that

$$1155 \quad \begin{aligned} \sum_{\ell=1}^k \theta_\ell &= \sum_{\ell=1}^{k_1 m + k_2} \theta_\ell = \sum_{i=0}^{k_1-1} \left(\sum_{j=1}^m \theta_{im+j} \right) + \sum_{j=1}^{k_2} \theta_{k_1 m + j} + 0 \\ &= \sum_{i=0}^{k_1-1} \left(\sum_{j=1}^m \theta_{im+j} \right) + \sum_{i=k_1}^{k_1} \left(\sum_{j=1}^{k_2} \theta_{im+j} \right) + \sum_{i=k_1+1}^{n-1} \left(\sum_{j=1}^0 \theta_{im+j} \right) \\ &= \sum_{i=0}^{k_1-1} \left(\sum_{j=1}^{\phi_{3,i}(k)} \theta_{im+j} \right) + \sum_{i=k_1}^{k_1} \left(\sum_{j=1}^{\phi_{3,i}(k)} \theta_{im+j} \right) + \sum_{i=k_1+1}^{n-1} \left(\sum_{j=1}^{\phi_{3,i}(k)} \theta_{im+j} \right) \\ &= \sum_{i=0}^{n-1} \left(\sum_{j=1}^{\phi_{3,i}(k)} \theta_{im+j} \right) \end{aligned} \quad (21)$$

1156 for $k \in \{1, 2, \dots, nm\}$, where the second to last equality comes from Equation (20). It is easy to verify
 1157 that Equation (21) also holds for $k = 0$, i.e.,

$$1158 \quad \sum_{\ell=1}^0 \theta_\ell = 0 = \sum_{i=0}^{n-1} \left(\sum_{j=1}^0 \theta_{im+j} \right) = \sum_{i=0}^{n-1} \left(\sum_{j=1}^{\phi_{3,i}(0)} \theta_{im+j} \right).$$

1159 Therefore, we have

$$1160 \quad \sum_{\ell=1}^k \theta_\ell = \sum_{i=0}^{n-1} \left(\sum_{j=1}^{\phi_{3,i}(k)} \theta_{im+j} \right) \quad \text{for any } k \in \{0, 1, \dots, nm\}. \quad (22)$$

1161 Fix $i \in \{0, 1, \dots, n-1\}$. By setting $p = \phi_{3,i}(k) \in \{0, 1, \dots, m\}$ and $\xi_j = \theta_{im+j}$ for $j = 1, 2, \dots, m$ in
 1162 Equation (16), we have

$$1163 \quad g(\phi_{3,i}(k) + \text{bin}0.\theta_{im+1}\theta_{im+2}\cdots\theta_{im+m}) = \sum_{j=1}^{\phi_{3,i}(k)} \theta_{im+j}. \quad (23)$$

1164 With Equations (17), (18), (19), (22), and (23) in hand, we are ready to construct the desired function
 1165 ϕ , which can be realized by the NestNet in Figure 22. Clearly, we have

$$1166 \quad \phi(k + \text{bin}0.\theta_1\cdots\theta_{nm}) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 0, 1, \dots, nm.$$

1167 Note that $nm = n \cdot n^r = n^{r+1}$. Then we have

$$1168 \quad \phi(k + \text{bin}0.\theta_1\cdots\theta_{n^{r+1}}) = \sum_{\ell=1}^k \theta_\ell \quad \text{for } k = 0, 1, \dots, n^{r+1}.$$

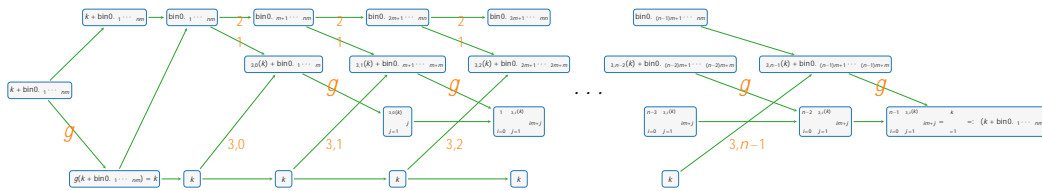


Figure 22: An illustration of the NestNet realizing ϕ based on Equations (17), (18), (19), (22), and (23). Here, g and \tilde{g} are regarded as activation functions.

1169 It remains to estimate the number of parameters in the NestNet realizing ϕ . Recall that ϕ_1 can
 1170 be realized by a one-hidden-layer \tilde{g} -activated network of width 1 and ϕ_2 can be realized by a
 1171 one-hidden-layer (σ, \tilde{g}) -activated network of width 2.

1172 Observe that

$$1173 \quad \min\{a, b\} = \frac{1}{2}(\sigma(a+b) - \sigma(-a-b) - \sigma(a-b) - \sigma(-a+b)) \quad \text{for any } a, b \in \mathbb{R}.$$

1174 As we can see from Figure 23, $\phi_{3,i}$ can be realized by a σ -activated network of width 4 and depth 2
 1175 for each $i \in \{0, 1, \dots, n-1\}$.

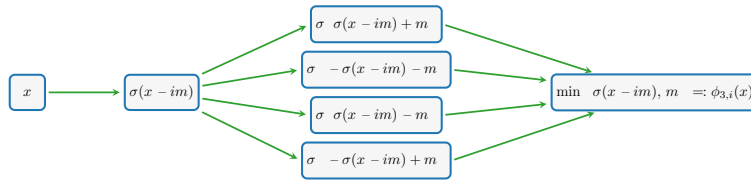


Figure 23: An illustration of $\phi_{3,i}$ for each $i \in \{0, 1, \dots, n-1\}$.

1176 Thus, the network in Figure 22 can be regarded as a (σ, g, \tilde{g}) -activated network of width $2 + 1 + 1 +$
 1177 $1 + 4 + 1 = 10$ and depth $2 + (2 + 1)n = 3n + 2$. Recall that $g \in \mathcal{NN}_r\{\widehat{n}\}$ and $\tilde{g} \in \mathcal{NN}_r\{72(r+7)n\}$.
 1178 This implies that ϕ can be realized by a height- $(r+1)$ NestNet with at most

$$1179 \quad \underbrace{(10+1)10((3n+2)+1)}_{\text{outer network}} + \underbrace{\widehat{n}}_g + \underbrace{72(r+7)n}_{\tilde{g}} \leq \widehat{n} + 114(r+7)(n+1)$$

1180 parameters, which means we finish the proof of Lemma D.5.